Calculus — Homework 1 (Fall 2025)

1. Let $\mathbb{N}$ denote the set of positive integers. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map, i.e.,

   (i) $f(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$;

   (ii) if $m < n$, then $f(m) < f(n)$.

   Prove that $f(n) \geq n$ for all $n \in \mathbb{N}$. (Hint: Use mathematical induction.)

2. Let $(a_n)_{n=1}^{\infty}$ be the sequence satisfying the given rules. Find an explicit formula for a_n that does not involve an recursive relation. Prove your answer.

   (a) $a_1 = 1;\qquad a_{n+1} = \frac{1}{2}a_n + 1.$

   (b) $a_1 = 1;\qquad a_{n+1} = \frac{n}{n+1}a_n.$

   (c) $a_1 = 1;\qquad a_{n+1} = a_n + \frac{1}{n(n+1)}.$

   (d) $a_1 = 1;\qquad a_{n+1} = a_n + \cdots + a_1.$

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence, and $L \in \mathbb{R}$. Prove that $\lim_{n\to\infty} a_n = L$ if and only if $\lim_{n\to\infty} |a_n - L| = 0$.

4. Let $(a_n)_{n=1}^{\infty}$ be a sequence, and $L \in \mathbb{R}$.

   (a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.

   (b) Is there a divergent sequence $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |b_n| = L$? Prove your answer. (Hint: Consider two cases: $L = 0$ and $L \neq 0$.)

5. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences, and $\gamma \in \mathbb{R}$. Prove that

   (a) $\lim_{n\to\infty}(\gamma \cdot a_n) = \gamma \cdot \lim_{n\to\infty} a_n$;

   (b) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$;

   (c) if $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

6. Compute $\lim_{n\to\infty} a_n$.

   (a) $a_n = \frac{(-1)^n}{n}$.

   (b) $a_n = \frac{n + (-1)^n}{n}$.

   (c) $a_n = \frac{2^n}{4^n + 1}$.

   (d) $a_n = \frac{1}{2n} - \frac{1}{2n+3}$.

   (e) $a_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$.

   (f) $a_n = \sqrt{n^2 + n} - \sqrt{n^2 - n}$.