Calculus — Homework 13 (Fall 2025)

1. Calculate.

(a) $\displaystyle\int_0^{\pi/6} \sin^2 3x\,dx.$

(b) $\displaystyle\int_0^{\pi} \sin^3 x\,dx.$

(c) $\displaystyle\int \sin^3 x\cos^3 x\,dx.$

(d) $\displaystyle\int \sin^2 x\cos^4 x\,dx.$

2. Use integration by parts to show that for $n > 2$,

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

Then compute

$$\int_0^{\pi/2} \sin^{2k} x\,dx \qquad \text{and} \qquad \int_0^{\pi/2} \sin^{2k+1} x\,dx$$

for $k \geq 1$.

3. Calculate.

(a) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{16-x^2}}\,dx.$

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$

(c) $\displaystyle\int \frac{x^2}{\sqrt{x^2-4}}\,dx.$

(d) $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\,dx.$

(e) $\displaystyle\int \frac{x}{\sqrt{x^2-2x+3}}\,dx.$

(f) $\displaystyle\int \frac{1}{(x^2+1)^3}\,dx.$

4. Decompose the rational functions into the form

$$p(x) + \frac{q_1(x)}{(x-b_1)^{r_1}} + \cdots + \frac{q_i(x)}{(x-b_i)^{r_i}} + \frac{q_{i+1}(x)}{(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}} + \cdots + \frac{q_j(x)}{(x^2+c_jx+d_j)^{r_j}},$$

where $p(x), q_k(x)$ are polynomials, and the degree of $q_k(x)$ is smaller than the degree of denominator.

(a) $\displaystyle\frac{1}{x^2+7x+6}.$

(b) $\displaystyle\frac{x}{x^4-1}.$

(c) $\displaystyle\frac{x^5+x^2}{x^4-1}.$

(d) $\displaystyle\frac{x^4}{x^3+x^2-2x}.$

5. Calculate.

(a) $\displaystyle\int \frac{1}{x^2+7x+6}\,dx.$

(b) $\displaystyle\int \frac{x^5}{(x-2)^2}\,dx.$

(c) $\displaystyle\int \frac{1}{x^2-6x+13}\,dx.$

(d) $\displaystyle\int \frac{x}{x^2-6x+13}\,dx.$

(e) $\displaystyle\int \frac{1}{(x^2-2x+2)^2}\,dx.$

(f) $\displaystyle\int \frac{x}{(x^2-2x+2)^2}\,dx.$

(g) $\displaystyle\int \frac{x}{x^4-1}\,dx.$

(h) $\displaystyle\int \frac{x^5+x^2}{x^4-1}\,dx.$

(i) $\displaystyle\int \frac{x^4}{x^3+x^2-2x}\,dx.$

(j) $\displaystyle\int \frac{1}{(x-1)(x^2+1)^2}\,dx.$

6. Evaluate the integrals.

(a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

(b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

(c) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

(d) $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

(e) $\displaystyle\int_0^1 x\ln x\,dx$.

(f) $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+5x+6}$.

7. This problem shows that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ and $\displaystyle\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$ are different.

   (a) Show that $\displaystyle\int_0^\infty \frac{2x}{x^2+1}\,dx$ diverges and hence that $\displaystyle\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges.

   (b) Show that
   $$\lim_{b\to\infty}\int_{-b}^b \frac{2x}{x^2+1}\,dx = 0.$$

8. Does the integral converge or diverge? Prove your answers.

   (a) $\displaystyle\int_{1/2}^2 \frac{dx}{x\ln x}$.

   (b) $\displaystyle\int_0^{\pi/2} \tan x\,dx$.

   (c) $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$.

   (d) $\displaystyle\int_\infty^1 \frac{\sqrt{x+1}}{x^2}\,dx$.

   (e) $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$.

   (f) $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$.

9. Suppose $f \in C^1(a,b)$ satisfies the equation
   $$f'(x) = xf(x), \qquad \forall x \in (a,b).$$
   Prove that there exists $C \in \mathbb{R}$ such that $f(x) = Ce^{\frac{x^2}{2}}$ for all $x \in (a,b)$.