Calculus — Homework 12 (Fall 2025)

1. Let I and J be open intervals, and let $f : I \to J$ be an invertible differentiable function. Let $a \in I$ and set $b = f(a)$. Prove that if $f'(a) \neq 0$, then f^{-1} is differentiable at b. (Hint: See Theorem B.3.2 in the textbook.)

2. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are two continuous functions such that
$$f(x) = g(x), \qquad \forall x \in \mathbb{Q}.$$
Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

3. Evaluate.

   (a) $\displaystyle\int_1^e \frac{dx}{x}$.

   (b) $\displaystyle\int_e^{e^2} \frac{dx}{x}$.

   (c) $\displaystyle\int_4^5 \frac{x}{x^2-1}\,dx$.

   (d) $\displaystyle\int_{1/4}^{1/3} \tan(\pi x)\,dx$.

   (e) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$.

   (f) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

   (g) $\displaystyle\int_1^2 \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$.

   (h) $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$.

   (i) $\displaystyle\int_0^1 \frac{4-e^x}{e^x}\,dx$.

   (j) $\displaystyle\int_0^1 \frac{e^x}{4-e^x}\,dx$.

   (k) $\displaystyle\int_0^1 x(e^{x^2}+2)\,dx$.

   (l) $\displaystyle\int_0^1 \frac{4}{\sqrt{e^x}}\,dx$.

   (m) $\displaystyle\int_1^2 x10^{1+x^2}\,dx$.

   (n) $\displaystyle\int_1^5 \frac{\log_5 x}{x}\,dx$.

4. Evaluate.

   (a) $\displaystyle\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$.

   (b) $\displaystyle\lim_{x\to 0^+} x^x$.

5. Differentiate.

   (a) $f(x) = 3^{2x}$.

   (b) $f(x) = 2^{5x}3^{\ln x}$.

   (c) $f(x) = \sqrt{\log_3 x}$.

   (d) $f(x) = x^x$.

   (e) $f(x) = (\ln x)^x$.

   (f) $f(x) = \cos(2^x)$.

6. Prove that, if n is a positive integer, then there exists N such that
$$e^x > x^n$$
for all $x \geq N$.

7. Let $a < b$. Prove that
$$e^{\lambda a + (1-\lambda)b} \leq \lambda e^a + (1-\lambda)e^b$$
for any $\lambda \in [0, 1]$.

8. Let
$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$
Is f differentiable at $x = 0$? Is f twice differentiable at $x = 0$? Justify your answers.

9. Differentiate.

(a) $f(x) = \arctan(x+1)$.

(b) $f(x) = e^x \arcsin x$.

10. Let $a > 0$. Calculate
$$\int \frac{1}{a^2 + (x+b)^2}\,dx.$$

11. Verify the identity
$$\cosh(x+y) = \cosh x\,\cosh y + \sinh x\,\sinh y\,.$$

12. Calculate.

(a) $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx.$

(b) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{4-x^2}}\,dx.$

(c) $\displaystyle\int_{0}^{3/2} \frac{dx}{9+4x^2}.$

(d) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx.$

(e) $\displaystyle\int_{0}^{\ln 2} \sinh 2x\,dx.$

(f) $\displaystyle\int_{0}^{\ln 2} x\sinh x\,dx.$

(g) $\displaystyle\int_{0}^{1} \frac{dx}{5^x}.$

(h) $\displaystyle\int_{0}^{1} \frac{x^3}{1+x^4}\,dx.$

(i) $\displaystyle\int_{1}^{2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

(j) $\displaystyle\int_{-3}^{-2} \frac{dx}{x^2+6x+10}.$

(k) $\displaystyle\int_{0}^{1} xe^{-x}\,dx.$

(l) $\displaystyle\int_{0}^{1} x^3 e^{-x^2}\,dx.$

(m) $\displaystyle\int_{1}^{e^2} x\,\ln(\sqrt{x})\,dx.$

(n) $\displaystyle\int_{0}^{1/2} x\cos \pi x\,dx.$

(o) $\displaystyle\int_{0}^{1/4} \arcsin 2x\,dx.$

(p) $\displaystyle\int x^2(e^x-1)\,dx.$

(q) $\displaystyle\int (\ln x)^2\,dx.$

(r) $\displaystyle\int e^x \sin x\,dx.$

13. Show that if $f, g \in C^2[a,b]$ and $f(a) = g(a) = f(b) = g(b) = 0$, then
$$\int_a^b f(x)g''(x)\,dx = \int_a^b g(x)f''(x)\,dx.$$

14. Let $P_n(x)$ denote the **Legendre polynomial** of degree n, i.e.
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

(a) Suppose $n, m \in \mathbb{Z}$ and $n > m \geq 0$. Prove that $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$.

(b) Prove that $\displaystyle\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ for any $n \neq m$.