# Calculus — Homework 11 (Fall 2025)

**Remark.** *In the following problems, we assume that the domain is the maximal possible domain in the set of real numbers unless otherwise stated, and assume that the codomain is the range of the function.*

1. Find the area of the region bounded by the curves.

   (a) $y = \sqrt{x}, \qquad y = x^2.$

   (b) $y = 8, \qquad y = x^2 + 2x.$

2. Calculate.

   (a) $\displaystyle\int_0^1 x(x^2+1)^3\,dx.$

   (b) $\displaystyle\int_{-1}^0 3x^2(4+2x^3)^2\,dx.$

   (c) $\displaystyle\int_0^1 x\sqrt{x+1}\,dx.$

   (d) $\displaystyle\int_0^1 \frac{x+3}{\sqrt{x+1}}\,dx.$

   (e) $\displaystyle\int_{-\pi}^{\pi} \sin^4 x\cos x\,dx.$

   (f) $\displaystyle\int_0^1 \cos^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}x)\,dx.$

   (g) $\displaystyle\int_0^{\pi} x\cos x^2\,dx.$

   (h) $\displaystyle\int_0^{\pi} x^2\cos x\,dx.$

   (i) $\displaystyle\int_0^{\pi/2} \cos^2 2x\,dx.$

   (j) $\displaystyle\int_0^{2\pi} \sin^2 x\,dx.$

   (k) $\displaystyle\int_0^1 x(x+5)^{14}\,dx.$

   (l) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx.$

   (m) $\displaystyle\int_0^{\pi/2} \cos(\sqrt{x})\,dx.$

3. Calculate.

   (a) $\displaystyle\frac{d}{dx}\left(\int_0^{1+x^2} \frac{dt}{\sqrt{2t+5}}\right).$

   (b) $\displaystyle\frac{d}{dx}\left(\int_{3x}^{1/x} \cos 2t\,dt\right).$

   (c) $\displaystyle\int_{-3}^{3} \frac{t^3}{1+t^2}\,dt.$

   (d) $\displaystyle\int_{-\pi/4}^{\pi/4} (x^2 - 2x + \sin x + \cos 2x)\,dx.$

4. Let $f$ be a continuous function, and $a, b, c$ be real numbers.

   (a) Show that
   $$\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx.$$

   (b) Show that, if $c \neq 0$,
   $$\frac{1}{c}\int_{ac}^{bc} f(x/c)\,dx = \int_a^b f(x)\,dx.$$

5. Let $f$ be continuous on $[-a, a]$.

   (a) Show that
   $$\int_{-a}^0 f(x)\,dx = \int_0^a f(-x)\,dx.$$

   (b) Show that
   $$\int_{-a}^a f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx.$$

6. Suppose that $f$ is continuous on $[a, b]$, $a < b$, and $\displaystyle\int_a^b f(x)\,dx = 0$. Prove that there is at least one number $c$ in $(a, b)$ for which $f(c) = 0$.

7. Let $f \in C[a,b]$. Prove that
$$\Big|\int_a^b f(x)\,dx\Big| \le \int_a^b |f(x)|\,dx.$$

8. Let $f, g \in C[a,b]$. Prove that
$$\Big|\int_a^b f(x)g(x)\,dx\Big| \le \sqrt{\int_a^b |f(x)|^2\,dx}\sqrt{\int_a^b |g(x)|^2\,dx}.$$

9. Let $f \in C^1[a,b]$ with $f(a) = 0$. Suppose that $|f(x)|$ attains its maximum value $M = |f(x_0)|$ at some point $x_0 \in (a,b)$. Prove that
   (a) $\displaystyle\int_a^b |f'(x)|\,dx \ge M$;
   (b) $\displaystyle(b-a)\int_a^b |f'(x)|^2\,dx \ge M^2$.

10. Verify that $f$ has an inverse and find $(f^{-1})'(c)$.
   (a) $f(x) = x^3 + 1,\ x \in (-\infty, \infty);\ c = 9$.
   (b) $f(x) = 1 - 2x - x^3,\ x \in (-\infty, \infty);\ c = 4$.
   (c) $f(x) = \sin x,\ -\frac{1}{2}\pi < x < \frac{1}{2}\pi;\ c = -\frac{1}{2}$.
   (d) $f(x) = \dfrac{x+3}{x-1},\ x > 1;\ c = 3$.
   (e) $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\,dt,\ x \in (-\infty, \infty);\ c = 0$.

11. Set
$$f(x) = \int_2^x \sqrt{1+t^2}\,dt.$$
   (a) Show that $f$ has an inverse.
   (b) Find $(f^{-1})'(0)$.

12. Estimate the values.
   (a) $\ln 20$.
   (b) $\ln 16$.
   (c) $\ln 3^4$.

   ($\ln 2 \approx 0.7$, $\ln 3 \approx 1.1$, $\ln 5 \approx 1.6$.)

13. Show that
$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$
   (Hint: Note that $\frac{\ln x}{x-1} = \frac{\ln x - \ln 1}{x-1}$ and interpret the limit as a derivative.)

14. Determine the domain and find the derivative.
   (a) $f(x) = \ln 4x$.
   (b) $f(x) = (\ln x)^3$.
   (c) $f(x) = \ln(\ln x)$.
   (d) $f(x) = \dfrac{1}{\ln x}$.
   (e) $f(x) = \ln\left|\dfrac{x+2}{x^3-1}\right|$.
   (f) $f(x) = \sin(\ln x)$.

15. Prove that for $x > 0$,
$$x - \frac{x^2}{2} < \ln(1+x) < x.$$