Calculus — Homework 10 (Fall 2025)

1. Calculate the following limits.

(a) $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$.

(b) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$.

(c) $\lim_{x\to 0} \frac{x+\sin(\pi x)}{x-\sin(\pi x)}$.

(d) $\lim_{x\to 0} \frac{\cos x - \cos 3x}{\sin(x^2)}$.

(e) $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x}$.

(f) $\lim_{x\to 1} \frac{x^{1/2}-x^{1/4}}{x-1}$.

(g) $\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{\sin^2 x}$.

(h) $\lim_{x\to\frac{\pi}{2}} (x-\frac{\pi}{2}) \sec x$.

2. Read the descriptions about upper (Riemann) sum and lower (Riemann) sum on the following websites:

- upper Riemann sum
- lower Riemann sum

3. Show that

$$\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1.$$

4. Find the critical point(s) for

$$F(x) = \int_0^x \frac{t-1}{1+t^2}\,dt.$$

At each critical point, determine whether F has a local maximum, a local minimum, or neither.

5. Calculate $F'(x)$.

(a) $F(x) = \int_0^{x^3} t\cos t\,dt$.

(b) $F(x) = \int_{x^2}^1 (t-\sin^2 t)\,dt$.

6. Evaluate the integral.

(a) $\int_0^1 (2x-3)\,dx$.

(b) $\int_1^2 (2x+x^2)\,dx$.

(c) $\int_1^4 2\sqrt{x}\,dx$.

(d) $\int_0^1 (x^{3/4}-2x^{1/2})\,dx$.

(e) $\int_0^1 (x+1)^{17}\,dx$.

(f) $\int_1^2 \frac{6-x}{x^3}\,dx$.

(g) $\int_1^2 2x(x^2+1)\,dx$.

(h) $\int_0^{\pi/2} \cos x\,dx$.

(i) $\int_0^{2\pi} \sin x\,dx$.

(j) $\int_{-\pi/6}^{\pi/6} \sin x\cos x\,dx$.

(k) $\int_2^5 (x-3)\,dx$.

(l) $\int_2^5 |x-3|\,dx$.

7. Let f be a function such that f' is continuous on $[a,b]$. Show that

$$\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\Big[\big(f(b)\big)^2 - \big(f(a)\big)^2\Big].$$

8. (a) Prove that $x^6 - \frac{x^5}{5} + 1 > 0$ for all $x \in [0,1]$.

(b) Suppose a continuous function $f \in C[0,1]$ satisfies the following conditions:

(i) $\int_0^1 f(x) \cdot x^{10}\, dx = 5$;

(ii) $\int_0^1 f(x) \cdot (x^6 - \frac{x^5}{5} + 1)\, dx = -6$.

Prove that there exists $c \in [0,1]$ such that $f(c) = 0$.

9. Prove the following inequalities:

   (a) $\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}}\, dx \leq \frac{1}{10}$.

   (b) $\frac{11}{24} \leq \int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx \leq \frac{11}{12\sqrt{3}}$. (Hint: $\sqrt{1-x^2} = \frac{1-x^2}{\sqrt{1-x^2}}$ for $x \in [0, \frac{1}{2}]$.)

10. Prove that $\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\, dx = 0$.

11. Suppose that $f \in C(-\infty, \infty)$, and $f(x + 2\pi) = f(x)$ for all $x \in (-\infty, \infty)$. Prove that

$$\int_a^{a+2\pi} f(x)\, dx = \int_0^{2\pi} f(x)\, dx,$$

for all $a \in \mathbb{R}$.

12. Let $f \in C[0,a]$ and $a > 0$. Prove the following:

    (a) $\int_0^a f(x)\, dx = \int_0^a f(a-t)\, dt$.

    (b) $\int_0^a x f(x)\, dx = \frac{a}{2} \int_0^a f(x)\, dx$, if $f(a-x) = f(x)$ for all $x \in [0,a]$.