Calculus, Fall 2025, week 9

Recall

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

(i)

$\Updownarrow$

on $(a,b]$ $\Leftrightarrow$ (i)+(iii) on (a,b)

Let f be a function defined on $[a,b]$

on $[a,b)$ $\Leftrightarrow$ (i)+(ii)

We say f is differentiable on $[a,b]$ if

(i) f is differentiable at any point in (a,b)

(ii) $\displaystyle\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ exists.

(iii) $\displaystyle\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ exists.

Notations:

Let $I = (a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$.

where an endpoint a or b may be $\pm\infty$ if it lies on the open side.

① $C^0(I) = C(I)$

$= \{$ all the continuous functions on I $\}$

($f \in C^0(I) \Leftrightarrow f$ is continuous on I)

② $C^1(I)$

$= \{ f \in C^0(I) \mid$ f is differentiable on I and $f' \in C^0(I) \}$

③ $C^2(I)$

$= \{ f \in C^1(I) \mid f' \in C^1(I) \}$

$\vdots$

$C^k(I)$

$f^{(k)} \in C^0(I)$

$\Updownarrow$

$= \{ f \in C(I) \mid f \in C(I) \}$

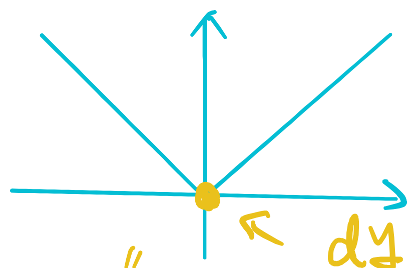
$\oplus$ $C^{\infty}(I) = \bigcap_{k=1}^{\infty} C^{k}(I)$

$= \left\{ f \;\middle|\; \begin{array}{l} \text{all } f^{(k)}(x) = \frac{d^k f}{dx^k} \text{ exist} \\ \text{and are continuous} \end{array} \right\}$

We say f is smooth on I if $f \in C^{\infty}(I)$

Recall

$y = |x|$



"corner"

$\left.\frac{dy}{dx}\right|_{x=0}$ does NOT exist

Thm (Chain Rule)

If g is differentiable at a and

f 〃 $g(a)$,

then $f \circ g$ is differentiable at a

and $\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{dy}{dx}$ $z = (f \circ g)(x)$

$dx \qquad dy \qquad dx \qquad y = g(x)$

$$\underline{(f\circ g)'(a)} = \underline{f'(g(a))\cdot g'(a)}$$

pf

$$= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} \quad \| $$

idea: replace it by

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \overset{?}{=} \underline{\lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)}} \cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

$g(x)-g(a) = 0$?

Consider

$$\varphi(y) := \begin{cases} \dfrac{f(y)-f(g(a))}{y-g(a)} & \text{if } y \neq g(a) \\ f'(g(a)) & \text{if } y = g(a) \end{cases}$$

Note that ① φ is continuous at $g(a)$

② g is differentiable at a

$\Rightarrow$ g is continuous at a

(*)

① + ② $\Rightarrow$ $\boxed{\varphi\circ g \text{ is continuous at } a.}$

Note that $\forall\, x \neq a$,

$$\frac{f(g(x)) - f(g(a))}{x-a} = \varphi(g(x)) \cdot \frac{g(x)-g(a)}{x-a}$$

$\begin{cases} \text{case 1}: & g(x) \neq g(a), \quad \text{RHS} = \dfrac{f(g(x)) - f(g(a))}{\cancel{g(x)-g(a)}} \cdot \dfrac{\cancel{g(x)-g(a)}}{x-a} \\ \text{case 2}: & g(x) = g(a) \end{cases}$

$\Rightarrow \text{RHS} = f'(g(a)) \cdot \dfrac{g(x)-g(a)}{x-a} = 0 = \text{LHS}$

So

$$(f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x \to a} \left(\varphi(g(x)) \cdot \frac{g(x)-g(a)}{x-a} \right)$$

$\varphi(g(x)) \xrightarrow{(*)} \varphi(g(a))$

$\dfrac{g(x)-g(a)}{x-a} \to g'(a)$ by assumption

$$= \lim_{x \to a} \varphi(g(x)) \cdot \lim_{x \to a} \frac{g(x)-g(a)}{x-a}$$

$$= \varphi(g(a)) \cdot g'(a)$$

$$= f'(g(a)) \cdot g'(a) \qquad \#$$

Example

① $\frac{d}{dx}(x^2-1)^{100} = (f\circ g)'(x)$

$f(y) = y^{100}$, $g(x) = x^2-1$

$\Rightarrow f(g(x)) = (x^2-1)^{100}$

$f'(y) = 100\cdot y^{99}$

$g'(x) = 2x$

$\overset{\text{chain rule}}{=} f'(g(x))\cdot g'(x)$

$= 100\cdot(x^2-1)^{99}\cdot 2x$. #

② $\frac{d}{dx}\underbrace{(x-\frac{1}{x})^{-3}}_{\frac{dz}{dx}} \overset{\text{chain}}{=} \underbrace{-3\cdot(x-\frac{1}{x})^{-4}}_{\frac{dz}{dy}}\cdot\underbrace{(x-\frac{1}{x})'}_{\frac{dy}{dx}}$

$(x-\frac{1}{x})' = (1+\frac{1}{x^2})$

$y = x - \frac{1}{x}$

$z = y^{-3}$

$\frac{dz}{dy} = -3\cdot y^{-4} = -3(x-\frac{1}{x})^{-4}$

$\frac{dy}{dx} = 1-(\frac{-1}{x^2})$

③ $\frac{d}{dx}(\underbrace{(x+1)}_{y}{}^5+1)^4$

$y = x+1$

$z = y^5+1$

$u = z^4$

$= \frac{du}{dx}$

$4z^3$ $\quad 5y^4$ $\quad 1$

chain $\frac{du}{\ }\ \frac{dz}{\ }$ chain $\frac{du}{\ }\ \frac{dz}{\ }\ \frac{dy}{\ }$

$$\underset{\text{rule}}{=} \frac{\ldots}{dz}\cdot\frac{\ldots}{dx} \underset{\text{rule}}{=} \frac{du}{dz}\cdot\frac{dz}{dy}\cdot\frac{dy}{dx}$$

$$= 4\left((x+1)^5+1\right)^3 \cdot 5(x+1)^4 \cdot 1$$

#

④ Let
$$\begin{cases} y = 3u+1 \\ u = x^{-2} \\ x = 1-s \end{cases}$$

Find $\frac{dy}{ds}$.

Sol

$$\frac{dy}{ds} = \frac{dy}{dx}\cdot\frac{dx}{ds} = \overbrace{\frac{dy}{du}}^{3}\cdot\overbrace{\frac{du}{dx}}^{-2\cdot x^{-3}}\cdot\overbrace{\frac{dx}{ds}}^{-1}$$

$$= 6x^{-3} = 6(1-s)^{-3}$$

#

Thm ((3.6.1), (3.6.2))

$\sin x$, $\cos x \in C^{\infty}(\mathbb{R})$.

And

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

pf

Recall that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

$\sin x \cos h + \cos x \sin h$

$\parallel$

So

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\sin x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{\to 1} \right)$$

$$= \cos x$$

$\cos x \cos h - \sin x \sin h$

$\parallel$

and

$$(\cos x)' = \lim \quad \frac{\cos(x+h) - \cos x}{}$$

$$(\cos x)' = \lim_{h\to 0} \frac{\quad}{h}$$

$$= \lim_{h\to 0}\left(\cos x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\to 0} - \sin x \cdot \underbrace{\frac{\sin h}{h}}_{\to 1}\right)$$

$$= -\sin x \quad \#$$

Recall: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Cor

- $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{\overbrace{(\sin x)'}^{\cos x} \cdot \cos x - \sin x \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

- $$(\cot x)' = \left(\tan\left(\tfrac{\pi}{2} - x\right)\right)'$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \left.\frac{d \tan y}{dy}\right|_{y = \frac{\pi}{2} - x} \cdot \frac{d\left(\frac{\pi}{2} - x\right)}{dx}$$

$$= \sec^2\left(\frac{\pi}{2}-x\right) \cdot (-1)$$

$$= -\csc^2 x$$

- $$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-(\cos x)'}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$$

- $$(\csc x)' = \left(\sec\left(\frac{\pi}{2}-x\right)\right)' \overset{\text{chain}}{\underset{\text{rule}}{=}} -\sec\left(\frac{\pi}{2}-x\right)\cdot\tan\left(\frac{\pi}{2}-x\right)$$

$$= -\csc x \cot x$$

__Example__

$y = x^2+1$ $\quad$ $\sec y \cdot \tan y$

① $$\frac{d}{dx}\sec(x^2+1) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{d\sec y}{dy} \cdot \frac{dy}{dx}$$

$$= \sec(x^2+1)\cdot\tan(x^2+1) \cdot 2x \ .\ \#$$

② $$(\sin x^{\circ})' = \left(\sin\left(\frac{\pi}{180}\cdot x\right)\right)'$$

$$\overset{\text{chain}}{=} \cos\left(\frac{\pi}{\ }x\right) \cdot \frac{\pi}{\ }$$

rule $\quad \cos(\frac{\pi}{180}x) \quad \frac{\pi}{180}$

$$= \frac{\pi}{180} \cdot \cos x^\circ \quad \#$$

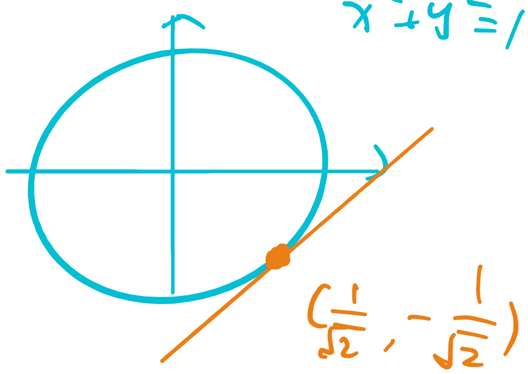
Implicit differentiation

Example

① Find the tangent line of

$$x^2 + y^2 = 1 \quad —(*)$$

at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.



sol

$$\frac{dy}{dx} = ?$$

Differentiate (*):

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0$$

$$\parallel$$

$$\underbrace{\frac{d}{dx}(x^2)}_{=2x} + \frac{d}{dx}(y^2) \overset{\text{chain rule}}{=} \underbrace{\frac{dy^2}{dy}}_{=2y} \cdot \frac{dy}{dx}$$

$$\parallel$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

At $(x,y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$,

$$2 \cdot \frac{1}{\sqrt{2}} + 2\left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = +1$$

$\Rightarrow$ The tangent line is

$$y + \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right) \qquad \#$$

② Suppose

$$\cos(x-y) = xy \qquad (**)$$

Q: $\frac{dy}{dx} = ?$

<u>Sol</u>

$(**) \Rightarrow$

$$\frac{d}{dx}\left(\cos(x-y)\right) = \frac{d}{dx}(xy) = \underbrace{(x)'}_{}\overset{1}{} \cdot y + x\overset{\frac{dy}{dx}}{(y)'}$$

$$\boxed{= y + x \cdot \frac{dy}{dx}}$$

chain rule $\quad \| \quad z = x-y$

$d\cos z \qquad dz$

$$\frac{d\cos z}{dz} \cdot \frac{dz}{dx}$$

$$\shortparallel$$

$$-\sin z \cdot \left(1 - \frac{dy}{dx}\right)$$

$$\shortparallel$$

$$-\sin(x-y) \cdot \left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} \cdot (x - \sin(x-y)) = -\sin(x-y) - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(x-y) + y}{\sin(x-y) - x} \quad \#$$

<u>Prop</u>

For $x > 0$, $\quad (n, m \in \mathbb{Z},\ n \neq 0)$

$$\left(x^{\frac{m}{n}}\right)' = \frac{m}{n} \cdot x^{\frac{m}{n}-1}$$

pf

<u>Step 1</u>: $\left(x^{\frac{1}{n}}\right)' = ?$

Let $y = x^{\frac{1}{n}}$

$$\Rightarrow y^n = x \Rightarrow \frac{d}{dx}(y^n) = \frac{dx}{dx} = 1$$

chain rule

$$\frac{dy^n}{dy} \cdot \frac{dy}{dx} \quad \ldots n-1 \quad \frac{dy}{dx}$$

$$\frac{dy^n}{dy}\cdot\frac{dy}{dx} = n\cdot y^{n-1}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{n\cdot y^{n-1}} = \frac{1}{n\,(x^{\frac{1}{n}})^{n-1}}$$

$$= \frac{1}{n}\cdot x^{-\frac{n-1}{n}} = \frac{1}{n}\, x^{\frac{1}{n}-1}$$

Step 2:

$$(x^{\frac{m}{n}})' = \left((x^{\frac{1}{n}})^m\right)'$$

$y = x^{\frac{1}{n}}$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{d\,y^m}{dy}\cdot\frac{dy}{dx} = m\,y^{m-1}\,(x^{\frac{1}{n}})'$$

$$\overset{\text{by Step1}}{=} m\cdot(x^{\frac{1}{n}})^{m-1}\cdot\frac{1}{n}\cdot x^{(\frac{1}{n}-1)}$$

$$= \frac{m}{n}\cdot x^{\frac{m-\cancel{1}}{n}+\frac{\cancel{1}}{n}-1} = \frac{m}{n}\, x^{\frac{m}{n}-1}$$

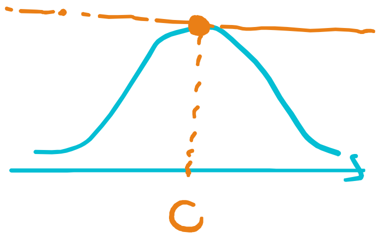
#

平均

§ Mean value theorem 均值定理

Lemma

Let f be a differentiable function on (a, b). Suppose f takes a maximum value or a minimum value at $x = c$ $\in (a,b)$.



Then $f'(c) = 0$.

pf

(min is similar)

Assume f takes a max. value at $x=c$.

$\Rightarrow \quad f(c+h) - f(c) \le 0 \quad \forall\ c+h \in (a,b)$

$$\Rightarrow \quad \frac{f(c+h)-f(c)}{h} \begin{cases} \le 0 & \text{if } h>0 \quad ① \\ \ge 0 & \text{if } h<0 \quad ② \end{cases}$$

$$f'(c) \ \overset{\text{by assumption } f'(c) \text{ exists}}{=}$$

$$① \Rightarrow \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$$

② $\Rightarrow f'(c) = \lim_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h} \geq 0$

$\Rightarrow \quad f'(c) = 0$

<u>Thm</u> (Mean Value Thm, Thm 4.1.1)

If f is differentiable on (a, b)

and continuous on $[a, b]$, then

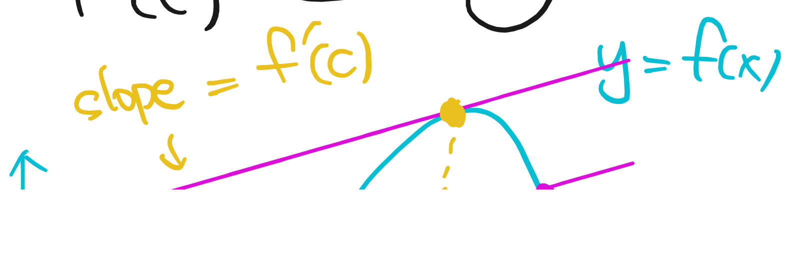
$\exists \; c \in (a, b)$ s.t.

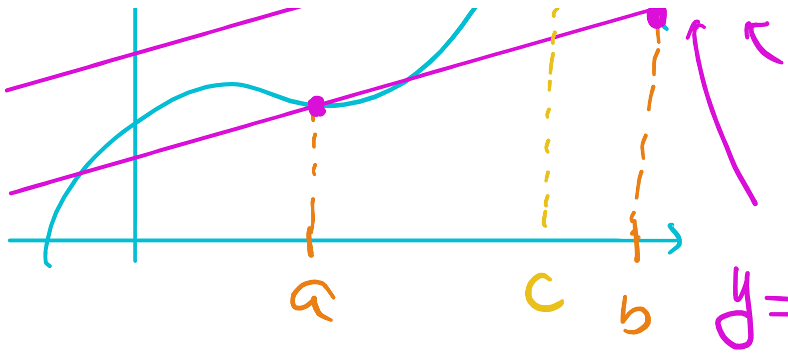
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In particular (Rolle's Thm, Thm 4.1.3),
if, in addition, $f(a) = f(b) = 0$, then

$\exists \; c \in (a, b)$ s.t.

$$f'(c) = 0$$





$$\text{slope} = \frac{f(b)-f(a)}{b-a}$$

$$y = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

pf

Step1 (Rolle's Thm):

Suppose $g(a) = g(b) = 0$, and g is differentiable on (a,b), continuous on $[a,b]$

By the extreme value thm, $\exists\ c_0, c_1 \in [a,b]$ such that g takes a minimum value $\underbrace{\ }_{m=g(c_0)}$ at c_0

" maximum " $\underbrace{\ }_{M=g(c_1)}$ c_1

① If $m = M = 0$, then $g(x) = 0 \quad \forall x \in [a,b]$

$\Rightarrow g'(x) = 0 \quad \forall x \in (a,b)$

$\Rightarrow \exists c \left(\overset{e.g.}{=} \frac{a+b}{2}\right) \in (a,b)$ s.t. $g'(c) = 0$

② If not, i.e. one of m and M is nonzero assume $\overset{=g(c_1)}{M} \neq 0$ ($m \neq 0$ is similar).

$\Rightarrow c_1 \neq a, b \Rightarrow c_1 \in (a,b)$

By Lemma, $g'(c_1) = 0 \Rightarrow$ Rolle's Thm.

<u>Step2</u>(MVT):

Consider

$$g(x) = f(x) - \underbrace{\left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right)}_{\text{polynomial} \Rightarrow \text{diff. conti.}}$$

$$\Rightarrow \quad g(a) = f(a) - (\ 0 \ + f(a)) = 0$$

$$g(b) = f(b) - \left(\frac{f(b)-\cancel{f(a)}}{\cancel{b-a}} \cancel{(b-a)} + \cancel{f(a)} \right) = 0$$

g is differentiable on (a,b)

" continuous on $[a,b]$

$\Rightarrow$ By <u>Step1</u>, $\exists \ c \in (a,b)$ s.t.

$$g'(c) = 0$$

$$\|$$

$$f(b) - f(a)$$

$$f'(c) - \frac{\quad}{b-a}$$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \#$$

Cor (Generalized Mean Value Thm).

Let $f, g \in C^0[a,b] \cap C'(a,b)$

If $g'(x) \neq 0 \ \forall x \in (a,b)$, then $\exists c \in (a,b)$

s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$(b > a)$

pf

First note that by MVT, $\exists c_0$ s.t.

$$\frac{g(b)-g(a)}{b-a} = g'(c) \neq 0$$

$$\Rightarrow \quad g(b)-g(a) = g'(c) \cdot (b-a) \neq 0$$

Consider

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\left(g(x)-g(a)\right)$$

$\in C^0[a,b] \cap C^1(a,b)$

$\Rightarrow$ By MVT, $\exists\, c \in (a,b)$ s.t.

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$

$F(b) = f(b)-f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(b)-g(a)) = 0$

$F(a) = f(a)-f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(a)-g(a)) = 0$

$$= 0$$

$$F'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(c)$$

$$\Rightarrow \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad \#$$

<u>Example</u>

Let $f(x) = \sqrt{1-x}$, $\quad x \in [-1,1]$.

$\Rightarrow \quad f \in C^0[-1,1] \cap C^1(-1,1)$

By MVT, $\exists\, c \in (-1,1)$ s.t.

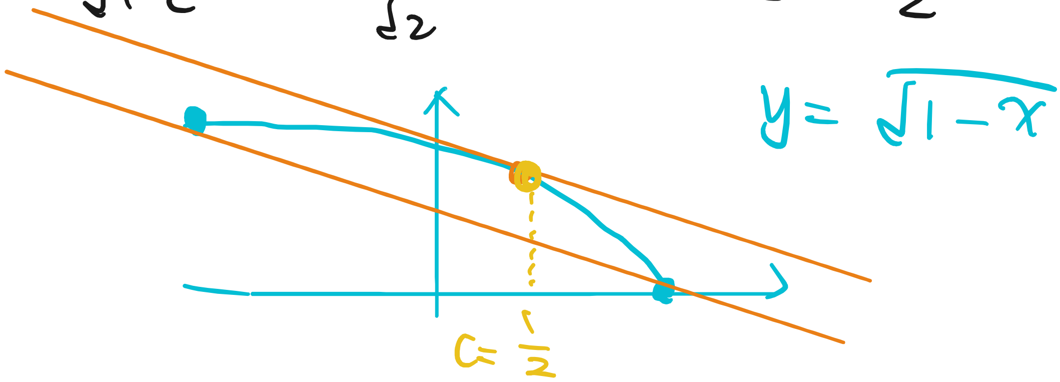
$$f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = -\frac{\sqrt{2}}{2}$$

In this example, we can solve c:

$$f'(c) = \frac{d}{dx}(1-x)^{\frac{1}{2}}\Big|_{x=c} = \frac{1}{2}(1-x)^{\frac{1}{2}-1}\cdot(-1)\Big|_{x=c}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{1-c}} = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \sqrt{1-c} = \frac{1}{\sqrt{2}} \Rightarrow 1-c = \frac{1}{2} \Rightarrow c = \frac{1}{2} \quad \#$$



<u>Example</u>

Suppose $f \in C^1(\mathbb{R})$. Assume

(i) $f(1) = 2$

(ii) $2 \le f'(x) \le 3 \quad \forall x \in \mathbb{R}$.

Prove that $8 \le f(4) \le 11$.

pf

By MVT, $\exists\, c \in (1,4)$ s.t.

$$2 \le f'(c) = \frac{f(4) - \overset{=2}{f(1)}}{\quad} \le 3$$

$2 \qquad\qquad 4-1$

By (i), (ii),

$$2 \leq \frac{f(4)-2}{3} \leq 3$$

$$\Rightarrow \underset{\underset{8}{\shortparallel}}{6+2} \leq f(4) \leq \underset{\underset{11}{\shortparallel}}{9+2}$$

#

e.g. $1-\frac{100}{n} \to 1 > 0$

$|a-b| \ge |a|-|b|$

$\shortparallel$

$|b-a| \ge |b|-|a|$

$|a-b| \ge \big||a|-|b|\big|$

常見錯誤

1.(a)

A. 好多人的寫法是這樣，主要是三角不等式只做了單邊:

$|a_n|-|L| \le |a_n - L| \implies ||a_n|-|L|| \le |a_n - L|$，應該要再做另一邊的不等式才能有如此推論。

B. 有少數比較誇張的是這樣:

分成3種情形討論，當然中間的論證完全不行。

If $L > 0$, then $a_n > 0$ for all $n \in \mathbb{N}$. $\implies ||a_n|-|L|| = |a_n - L|$...

~~If $L = 0$, then $a_n = 0$ for all $n \in \mathbb{N}$.~~ $\implies ||a_n|-|L|| = |0-0| = 0$...

~~If $L < 0$, then $a_n < 0$~~ for all $n \in \mathbb{N}$. $\implies ||a_n|-|L|| = |L-a_n| = |a_n - L|$...

e.g. $a_n = \frac{(-1)^n}{n} \to 0$

1.(b)

幾乎沒問題，有些人可能忘記討論$L < 0$的情形。

$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{L}$

2.

A. 有些人的錯誤是直接假設結論是對的:

$\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $n \ge N \implies |\frac{1}{b_n} - \frac{1}{L}| < \epsilon$. 接著推論一番，得到結論$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{L}$

B. ϵ並非固定，未必能找到N:

By the definition of $\lim_{n\to\infty} b_n = L$, $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $n \ge N \implies |b_n - L| < |b_n||L|\epsilon$. 接著對倒數估計...

ε

3.

大致上只有符號沒寫好:

例如: $\lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} n$，這樣子不顧極限是否存在使用limit law。

1 Problem 4

Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} a_n = \alpha$. Define the average sequence by

$\alpha-\epsilon$ $\alpha+\epsilon$

$$\sigma_n = \frac{a_1 + \cdots + a_n}{n}.$$

as $n\to\infty$ as $n\to\infty$

(a) Prove that for any $\epsilon > 0$, there exists N_ϵ such that for all $n \geq N_\epsilon$, the following inequality holds: e.g. $a_n = \frac{1}{n}$

$$\frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n-N_\epsilon)(\alpha-\epsilon)}{n} < \sigma_n < \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n-N_\epsilon)(\alpha+\epsilon)}{n}$$

- 同學們會寫

$\ldots, \alpha, \alpha, \alpha, \alpha, \alpha \ldots$

$$\text{錯誤寫法} \quad \lim_{n\to\infty} a_n = \alpha \implies \exists\, N \in \mathbb{N} \text{ such that } a_n = \alpha \text{ when } n > N.$$

(b) Prove that the sequence $(\sigma_n)_{n=1}^{\infty}$ is also convergent, and $\lim_{n\to\infty} \sigma_n = \alpha$.

- 本題無法直接應用夾擠定理 (pinching theorem)。比較精簡的做法應取 $\limsup$ 與 $\liminf$。
- 如果想要避免使用 sup 跟 inf，這裡提供一個寫法：(Sketch) 給定任意的 ϵ, 依照 a 小題可以找到 N_ϵ 使得

$$\sigma_n < \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n-N_\epsilon)(\alpha+\epsilon)}{n}$$

兩邊扣掉 α 後得到

$$\sigma_n - \alpha \leq \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n-N_\epsilon)(\alpha+\epsilon)}{n} - \alpha = \epsilon + \frac{1}{n}\underbrace{(a_1 + \cdots + a_{N_\epsilon} - N_\epsilon(\alpha+\epsilon))}_{M}$$

由於 M 是固定實數，我們可以找一個 $\tilde{N}$ 使得當 $n > \tilde{N}$ 時，$\frac{M}{n} < \epsilon$ 。因此對任意的 ϵ，我們都可以找到 $N = \max\{N_\epsilon, \tilde{N}\}$ 使得當 $n > N$ 時，$\sigma_n - \alpha < 2\epsilon$。不等式的另一側是類似的做法。

2 Problem 5

(f) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$.

- 應該把 x 看成**負**的，所以 $\sqrt{\frac{1}{x^2}} = -\frac{1}{x}$.

3 Problem 6

Let $f(x)$ be a real valued function which is continuous in $(-\infty, \infty)$. Suppose that

$$\lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1.$$

Prove that $f(x)$ has infinitely many zeros in the half line $(0, +\infty)$.

- 把題目誤解為函數的極限而非數列的極限。也就是說把 $n \to \infty$ 的 n 視為 $\mathbb{R}$ 在討論。
- 有些人會寫：

$$\text{錯誤寫法：} \quad \lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1 \implies f(n\pi + \frac{\pi}{3}) = \sin(n\pi + \frac{\pi}{3})$$

- 忽略了 $\sin(n\pi + \frac{\pi}{3})$ 會發散：

$$\frac{\lim_{n\to\infty} f(n\pi + \frac{\pi}{3})}{\lim_{n\to\infty} \sin(n\pi + \frac{\pi}{3})} \neq$$

diverges

$$\text{錯誤寫法：} \quad \lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1 \implies \lim_{n\to\infty} f(n\pi + \frac{\pi}{3}) = \lim_{n\to\infty} \sin(n\pi + \frac{\pi}{3})$$

- 忽略了當 n 是奇數時，$\sin(n\pi + \frac{\pi}{3}) < 0$ 而忘記變號：

$$\text{錯誤寫法：} \quad \frac{1}{2} < \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} < \frac{3}{2} \implies \frac{1}{2}\sin(n\pi + \frac{\pi}{3}) < f(n\pi + \frac{\pi}{3}) < \frac{3}{2}\sin(n\pi + \frac{\pi}{3})$$

$a < b$. $r<0 \Rightarrow ra > rb$

4 Problem 7

check: ① 分子 continuous
② 分母 ..
③ 分母 $\neq 0$

Let

$$f(x) = \begin{cases} \frac{\sin(x-\frac{\pi}{2}\cos x)}{x^2+1}, & \text{if } |x| > 2\pi, \\ -\frac{x^2}{4\pi^2(4\pi^2+1)}, & \text{if } |x| \le 2\pi. \end{cases}$$

$[-2\pi, 2\pi]$

$$\left|\frac{\sin(\cdots)}{x^2+1}\right| \le \frac{1}{x^2+1} < \frac{1}{(2\pi)^2+1} \quad \text{when } |x| > 2\pi$$

(a) Prove that f is continuous on $(-\infty, \infty)$.

- 忘記說明為何函數在 $(-\infty, -2\pi) \cup (-2\pi, +2\pi) \cup (+2\pi, +\infty)$ 是連續的。
- 忘記說明 $x^2 + 1 \neq 0$。這是重要的！
- 函數在 c 連續應包含兩件事：

$$1. \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad 2. \lim_{x \to c} f(x) = f(c)$$

同學大多忘記把條件 2 寫進過程了。

(b) Prove that f is bounded on $(-\infty, \infty)$, i.e. there exists $M > 0$ such that $|f(x)| \le M$ for any $x \in (-\infty, \infty)$.

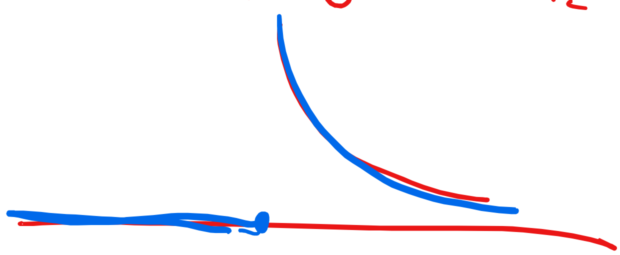
- 有部分同學寫到 $\lim_{x \to \pm\infty} f(x) = 0$ 表示 f 有界。這裡應該要加上條件 f 是連續的。

$(a_n)_{n=1}^{\infty}$: seq

$$\lim_{n \to \infty} a_n \text{ exists} \Rightarrow (a_n) \text{ is bdd}$$

$$\lim_{x \to \infty} f(x) \text{ exists} \not\Rightarrow f \text{ is bdd}$$

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x \le 0 \end{cases}$$



National Tsing Hua University

Calculus I – Midterm

Instructor: Hsuan-Yi Liao

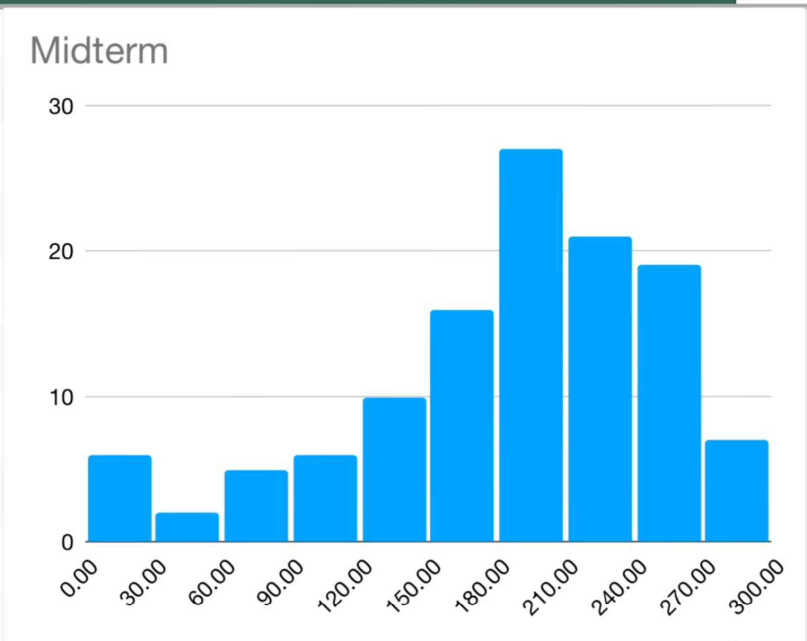
Fall, 2025

Name: Answer

Student ID: ______________________

- This exam contains 9 pages (including this cover page) and 8 questions.
- Total of points is 300.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.
- The use of L'Hôspital's rule is **NOT** allowed in this exam.

學號 中文姓名	Midterm	Quiz3	Quiz4	Quiz5	Final
Average	182.74789				
Average (except zero)	192.45132				
Quartile 0	0				
Quartile 1	150				
Quartile 2	196				
Quartile 3	237.5				
Quartile 4	296				
標準差	70.213502				
非零數	113				
不到50%人數	29				



1. Let $(a_n)_{n=1}^{\infty}$ be a sequence, and $L \in \mathbb{R}$.
   (a) (15 points) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.
   (b) (15 points) Is there a divergent sequence $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |b_n| = L$? Prove your answer. (Hint: Consider two cases: $L = 0$ and $L \neq 0$.)

(a) Since $\lim_{n\to\infty} a_n = L$, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t.

$$|a_n - L| < \varepsilon \quad \forall n \geq N_\varepsilon$$

$$\Rightarrow \quad \big||a_n| - |L|\big| \leq |a_n - L| < \varepsilon \quad \forall n \geq N_\varepsilon$$

$$\Rightarrow \quad \lim_{n\to\infty} |a_n| = |L|. \quad \#$$

(b) Case 1: $L = 0$.

If $\lim_{n\to\infty} |b_n| = 0$, then $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon$

$$\underbrace{\big||b_n| - 0\big|}_{\substack{\parallel \\ |b_n| = |b_n - 0|}} < \varepsilon$$

NO

$$\Rightarrow \lim_{n\to\infty} b_n = 0$$

Case 2: $L > 0$.

Yes

Choose $b_n = (-1)^n L \Rightarrow \lim_{n\to\infty} |b_n| = L$, but $(b_n)_{n=1}^{\infty}$ diverges (since it has 2 subseq's converging to different numbers: $b_{2k-1} \to -L$, $b_{2k} \to L$ as $k \to \infty$)

Case 3: $L < 0$.

Since $|b_n| \geq 0$ $\forall n$, $\lim_{n\to\infty} |b_n| \geq 0 \Rightarrow \nexists L < 0$. $\#$

NO

2. (30 points) Let $(b_n)_{n=1}^{\infty}$ be a convergent sequence such that $\lim_{n\to\infty} b_n \neq 0$. Use the definition of limits to prove that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}.$$

Let $\beta = \lim_{n\to\infty} b_n$,

$\forall \varepsilon > 0$, $\exists\ N_\varepsilon$ s.t. $|b_n - \beta| < \varepsilon \quad \forall n \geq N_\varepsilon$.

For $\varepsilon = \frac{|\beta|}{2} > 0$, $\exists\ \tilde{N}$ s.t. $|b_n - \beta| < \frac{|\beta|}{2} \quad \forall n \geq \tilde{N}$

$|b_n - \beta| \geq |\beta| - |b_n|$

$\Rightarrow |b_n| > |\beta| - \frac{|\beta|}{2} = \frac{|\beta|}{2} > 0 \quad \forall n \geq \tilde{N}$,

Note that

$$\left|\frac{1}{b_n} - \frac{1}{\beta}\right| = \frac{|\beta - b_n|}{|b_n \beta|} \leq \frac{|b_n - \beta|}{\frac{|\beta|}{2} \cdot |\beta|} \quad \forall n \geq \tilde{N}$$

Choose $N' = \max\left\{\tilde{N}, N_{\frac{|\beta|^2}{2}\cdot\varepsilon}\right\}$

$\Rightarrow \forall n \geq N'$,

$$\left|\frac{1}{b_n} - \frac{1}{\beta}\right| \leq \frac{|b_n - \beta|}{\frac{|\beta|^2}{2}} < \frac{1}{\frac{|\beta|^2}{2}} \cdot \frac{|\beta|^2}{2}\cdot\varepsilon = \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\beta} \quad \#$$

3. Prove the convergence of the sequence:

(a) (20 points) Prove that the sequence $\left(\frac{\sin(n)+\cos(n)}{\sqrt{n}}\right)_{n=1}^{\infty}$ converges.

(b) (20 points) Prove that the sequence $\left(\frac{n^4-3n^2+n+2}{n^3-7n}\right)_{n=1}^{\infty}$ diverges.

(c) (20 points) Prove that the sequence $\left(\cos\left(\frac{n\pi}{3}\right)\right)_{n=1}^{\infty}$ diverges.

(a) $$\left|\frac{\sin n+\cos n}{\sqrt{n}}\right| \le \frac{|\sin n|+|\cos n|}{\sqrt{n}} \le \frac{2}{\sqrt{n}} \to 0$$ as $n\to\infty$

By the pinching thm,

$\lim_{n\to\infty} \frac{\sin n+\cos n}{\sqrt{n}}$ exists, $= 0$. #

(b) $$\frac{n^4-3n^2+n+2}{n^3-7n} = \frac{n^4}{n^3-7n} - \frac{3n^2+n+2}{n^3-7n}$$

$$\frac{3n^2+n+2}{n^3-7n} = \frac{3\frac{1}{n}+\frac{1}{n^2}+\frac{2}{n^3}}{1-7\frac{1}{n^2}} \to 0$$ as $n\to\infty$

Convergent $\Rightarrow$ bdd

$$\frac{n^4}{n^3-7n} = \frac{n}{1-7\frac{1}{n^2}} \ge n \quad \forall n\ge 3$$

$\Rightarrow$ unbdd

$\Rightarrow$ $\frac{n^4-3n^2+n+2}{n^3-7n}$ is unbdd $\Rightarrow$ div. #

(c) $\exists$ 2 subseq. with different limits:

$$\cos\left(\frac{6k\pi}{3}\right) = 1 \to 1$$

$$\neq \quad \text{as } k\to\infty$$

$$\cos\left(\frac{(6k+1)\pi}{3}\right) = \frac{1}{2} \to \frac{1}{2}$$

$\Rightarrow$ div. #

$\exists M$ s.t. $\Rightarrow |a_n| \le M \quad \forall n$

4. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} a_n = \alpha$. Define the average sequence by

$$\sigma_n = \frac{a_1 + \cdots + a_n}{n}.$$

n time

$$|\sigma_n| \le \frac{|a_1| + \cdots + |a_n|}{n} \le \frac{M + \cdots + M}{n} = M$$

(a) (15 points) Prove that for any $\epsilon > 0$, there exists N_ϵ such that for all $n \ge N_\epsilon$, the following inequality holds:

$$\frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha - \epsilon)}{n} < \sigma_n < \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha + \epsilon)}{n}.$$

$\alpha - \varepsilon < a_n < \alpha + \varepsilon$

$\Updownarrow$

(b) (15 points) Prove that the sequence $(\sigma_n)_{n=1}^{\infty}$ is also convergent, and $\lim_{n\to\infty} \sigma_n = \alpha$.

(a) Since $\lim_{n\to\infty} a_n = \alpha$, $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $|a_n - \alpha| < \varepsilon \quad \forall n \ge N_\varepsilon$

$\alpha - \varepsilon <$ each term $< \alpha + \varepsilon$

$\Rightarrow \forall n \ge N_\varepsilon$

$$\sigma_n = \frac{(a_1 + \cdots + a_{N_\varepsilon}) + (a_{N_\varepsilon+1} + \cdots + a_n)}{n}$$

$(n-N_\varepsilon)$-terms $\qquad\qquad$ $(n-N_\varepsilon)$-terms

$$\frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(\alpha-\varepsilon) + \cdots + (\alpha-\varepsilon)}{n} \qquad \wedge \qquad \frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(\alpha+\varepsilon) + \cdots + (\alpha+\varepsilon)}{n}$$

$$\Rightarrow \frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(n-N_\varepsilon)(\alpha-\varepsilon)}{n} < \sigma_n < \frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(n-N_\varepsilon)(\alpha+\varepsilon)}{n} \quad \forall n \ge N_\varepsilon$$

#

(b) By (a), we have

$$\underline{\lim}_{n\to\infty}\left(\frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(n-N_\varepsilon)(\alpha-\varepsilon)}{n}\right) \le \overline{\lim}_{n\to\infty} \sigma_n \le \overline{\lim}_{n\to\infty}\left(\frac{a_1 + \cdots + a_{N_\varepsilon}}{n} + \frac{(n-N_\varepsilon)(\alpha+\varepsilon)}{n}\right)$$

$$\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\alpha - \varepsilon \qquad\qquad\qquad\qquad\qquad\qquad \alpha + \varepsilon$$

$\forall \varepsilon > 0$.

$$\Rightarrow \overline{\lim}_{n\to\infty} \sigma_n = \alpha$$

Similarly, $\underline{\lim}_{n\to\infty} \sigma_n = \alpha$

$$\Rightarrow \lim_{n\to\infty} \sigma_n = \alpha$$

#

5. Evaluate the limits.

continuous

(a) (10 points) $\lim_{x\to -1} |x|(x^4 - 3)$. $= |1| \cdot (1^4 - 3) = -2$ #

(b) (10 points) $\lim_{x\to 1} \frac{x^{10} - 1}{x^3 - 1}$. $= \lim_{x\to 1} \frac{x^9 + \cdots + 1}{x^2 + x + 1} = \frac{10}{3}$ #

(c) (10 points) $\lim_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} 2x - 1, & x \le 2 \\ x^2 - x, & x > 2. \end{cases}$ $= \lim_{x\to 2^+} x^2 - x = 2^2 - 2 = 2$ #

(d) (10 points) $\lim_{x\to \pi} \frac{\sin x}{x - \pi}$.

(e) (10 points) $\lim_{x\to 0} \frac{\sin 7x - \sin 5x}{\sin x}$.

(f) (10 points) $\lim_{x\to -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$.

(d) $= \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} -\frac{\sin y}{y} = -1$ #

$y = x - \pi$

(e) $= \lim_{x\to 0} \frac{7 \cdot \frac{\sin 7x}{7x} - \frac{\sin 5x}{5x} \cdot 5}{\frac{\sin x}{x}} = \frac{7-5}{1} = 2$ #

(f) $= \lim_{x\to -\infty} \frac{\frac{1}{(-x)}\sqrt{x^2+1}}{-1 - \frac{1}{x}} = \lim_{x\to -\infty} \frac{\sqrt{\frac{1}{x^2}(x^2+1)}}{-1 - \frac{1}{x}}$

$x < 0$

$= \lim_{x\to -\infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{-1 - \frac{1}{x}} = \frac{1}{-1}$

$= -1$ #

6. (30 points) Let $f(x)$ be a real valued function which is continuous in $(-\infty, \infty)$. Suppose that

$$\lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1.$$

Prove that $f(x)$ has infinitely many zeros in the half line $(0, +\infty)$.

For $\varepsilon = \frac{1}{2} > 0$, $\exists N > 0$ s.t.

$$\left| \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} - 1 \right| < \frac{1}{2} \quad \forall n \geq N$$

$$\Rightarrow \quad \frac{1}{2} < \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} < \frac{3}{2} \quad \forall n \geq N$$

$$\sin(n\pi + \tfrac{\pi}{3}) = (-1)^n \frac{\sqrt{3}}{2}$$

For $n = 2k$, $k = [\frac{N}{2}]+1, [\frac{N}{2}]+2, \dots,$

$$\frac{1}{2} < \frac{f(2k\pi + \frac{\pi}{3})}{\frac{\sqrt{3}}{2}} < \frac{3}{2} \Rightarrow f(2k\pi + \tfrac{\pi}{3}) > \frac{\sqrt{3}}{4} > 0$$

For $n = 2k+1$, $k = [\frac{N}{2}]+1, [\frac{N}{2}]+2, \dots,$

$$\frac{1}{2} < \frac{f(2k\pi + \pi + \frac{\pi}{3})}{-\frac{\sqrt{3}}{2}} < \frac{3}{2} \Rightarrow f(2k\pi + \tfrac{4\pi}{3}) < -\frac{\sqrt{3}}{4} < 0$$

Since f is continuous, by IVT, $\forall k \geq [\frac{N}{2}]+1$ $\exists\, x_k \in (2k\pi + \frac{\pi}{3}, 2k\pi + \frac{4\pi}{3})$ s.t. $f(x_k) = 0$.

$\Rightarrow f$ has ∞ zeros in $(0, \infty)$. #

7. Let

$$f(x) = \begin{cases} \dfrac{\sin(x - \frac{\pi}{2}\cos x)}{x^2+1}, & \text{if } |x| > 2\pi, \\ -\dfrac{x^2}{4\pi^2(4\pi^2+1)}, & \text{if } |x| \le 2\pi. \end{cases}$$

(a) (15 points) Prove that f is continuous on $(-\infty, \infty)$.

(b) (15 points) Prove that f is bounded on $(-\infty, \infty)$, i.e. there exists $M > 0$ such that $|f(x)| \le M$ for any $x \in (-\infty, \infty)$.

(a) Since $x - \frac{\pi}{2}\cos x$ and $\sin y$ are continuous on $\mathbb{R}$,

So is $\sin(x - \frac{\pi}{2}\cos x)$

Since x^2+1 is continuous and $\neq 0$ $\forall x \in \mathbb{R}$,

$\dfrac{\sin(x-\frac{\pi}{2}\cos x)}{x^2+1}$ is continuous on $\mathbb{R}$.

$\Rightarrow$ (1) f is continuous at x $\forall$ $|x| > 2\pi$

Since $-\dfrac{x^2}{4\pi^2(4\pi^2+1)}$ is a polynomial, it is continuous on $\mathbb{R}$

$\Rightarrow$ (2) f is conti. at x $\forall$ $|x| < 2\pi$.

At $x = 2\pi$, $\displaystyle\lim_{x\to 2\pi^+} f(x) = \frac{\sin(2\pi - \frac{\pi}{2}\cos 2\pi)}{(2\pi)^2+1} = -\frac{1}{4\pi^2+1}$

$\displaystyle = \lim_{x\to 2\pi^-} f(x) = -\frac{(2\pi)^2}{4\pi^2(4\pi^2+1)} = f(2\pi)$

$\Rightarrow$ (3) f is conti. at $x = 2\pi$.

At $x = -2\pi$, $\displaystyle\lim_{x\to -2\pi^-} f(x) = -\frac{1}{4\pi^2+1} = \lim_{x\to -2\pi^+} f(x) = f(-2\pi)$

$\Rightarrow$ (4) f is conti. at $x = -2\pi$

#

(b) Since f is conti. on $[-2\pi, 2\pi]$, $\exists M$ s.t. $|f(x)| \le M$ $\forall x \in [-2\pi, 2\pi]$

For $|x| > 2\pi$, $|f(x)| \le \dfrac{1}{(2\pi)^2+1}$.

Take $M' = \max\{M, \dfrac{1}{(2\pi)^2+1}\}$. $\Rightarrow |f(x)| \le M'$ $\forall x \in \mathbb{R}$.

#

8. Let $f(x)$ be a differentiable function.

   (a) (15 points) State the definition of $f'(x)$, the derivative of f.

   (b) (15 points) Suppose $f(x) = x^3$. Use the definition of derivatives to prove that $f'(x) = 3x^2$.

(a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ #

(b) $(x^3)' = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h\to 0} \frac{\cancel{((x+h)-x)}^{h}\,((x+h)^2 + (x+h)x + x^2)}{\cancel{h}}$

$= \lim_{h\to 0} \left((x+h)^2 + (x+h)x + x^2\right) = 3x^2$ #