Calculus, Fall 2025, week 6

<u>Recall</u>

① Let $f(x)$ be a function defined in (a,b) and $c \in (a,b)$.

$$\boxed{f \text{ is continuous at } x=c}$$

$$\Longleftrightarrow \begin{cases} \text{(i)} \ \lim\limits_{x\to c} f(x) \ \text{exists} \\ \text{(ii)} \ \lim\limits_{x\to c} f(x) = f(c) \end{cases}$$

$\Longleftrightarrow$ for any seq. $x_n \in (a,b)$ and $\lim\limits_{n\to\infty} x_n = c$, we have

$$\lim_{n\to\infty} f(x_n) = f(c) = f\left(\lim_{n\to\infty} x_n\right)$$

② f is continuous on $[a,b]$ if

(i) f is continuous at any $c \in (a,b)$

(ii) $\lim\limits_{x\to a^+} f(x) = f(a)$

(iii) $\lim f(x) = f(b)$

$\lim_{x \to b^-} f(x) = f(b)$

③ Polynomials, $\sin x$, $\cos x$, $|x|$, are continuous on $(-\infty, \infty)$

Application to limits

Thm

If $\lim_{x \to c} f(x) = b$ and g is continuous at $x = b$, then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(b)$$

pf

Recall:

① $\lim_{x \to c} f(x) = b \iff \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - b| < \varepsilon \ \forall \ 0 < |x - c| < \delta$

② g is Continuous at b $\Leftrightarrow$ $\lim_{y\to b} g(y) = g(b)$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \delta' > 0$ s.t. $|g(y) - g(b)| < \varepsilon$ $\forall$ $|y - b| < \delta'$

idea: plug in $y = f(x)$

$|f(x) - b| < \delta'$

③ Want to show: $\lim_{x\to c} g(f(x)) = g(b)$

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \tilde{\delta} > 0$ s.t.

$$|g(f(x)) - g(b)| < \varepsilon \quad \forall\ 0 < |x - c| < \tilde{\delta}$$

Since g is Continuous at b, we have for any $\varepsilon > 0$, $\exists$ $\delta' > 0$ s.t.

$$|g(y) - g(b)| < \varepsilon \qquad \forall\ |y - b| < \delta'$$

Since $\lim_{x\to c} f(x) = b$, for $\delta' > 0$, $\exists$ $\delta > 0$ s.t.

$$|f(x) - b| < \delta' \qquad \forall\ 0 < |x - c| < \delta$$

So (set $y = f(x)$)

$$|g(f(x)) - g(b)| < \varepsilon \quad \forall\ 0 < |x - c| < \delta$$

$\Rightarrow \quad \lim_{x \to c} g(f(x)) = g(b)$ #

<u>Example</u>

① $\lim_{x \to \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

by Thm, cos is conti.

$\overset{\text{by Thm}}{=} \cos\Big(\underbrace{\lim_{x \to \frac{\pi}{2}} \left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)}\Big)$

$\underbrace{}_{\displaystyle = \lim_{x \to \frac{\pi}{2}} 2x + \lim_{x \to \frac{\pi}{2}} \sin\left(\frac{3\pi}{2} + x\right)}$

$\parallel \leftarrow$ sin is conti.

$2 \cdot \frac{\pi}{2} + \sin\Big(\underbrace{\lim_{x \to \frac{\pi}{2}} \left(\frac{3\pi}{2} + x\right)}_{\frac{3\pi}{2} + \frac{\pi}{2} = 2\pi}\Big)$

$= \cos\left(2 \cdot \frac{\pi}{2} + \overset{0}{\overset{\parallel}{\sin 2\pi}}\right) = \cos \pi = -1$ #

② $\lim_{n \to \infty} \sin\left(\frac{\pi}{n}\right) \overset{\text{sin is conti.}}{=} \sin\left(\lim_{n \to \infty} \frac{\pi}{n}\right)$

$$= \sin 0 = 0 \quad \#$$

$\sqrt{x}$ is continuous on $[0, +\infty)$

③ $$\lim_{n\to\infty} \sqrt{\frac{2n-1}{n}} = \sqrt{\lim_{n\to\infty} \frac{(2n-1)\times\frac{1}{n}}{n\times\frac{1}{n}}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{2-\frac{1}{n}}{1}} = \sqrt{2} \quad \#$$

$|x|$ is continuous

④ $$\lim_{n\to\infty} \left|\frac{2n-1}{n}\right| = \left|\lim_{n\to\infty} \frac{2n-1}{n}\right|$$

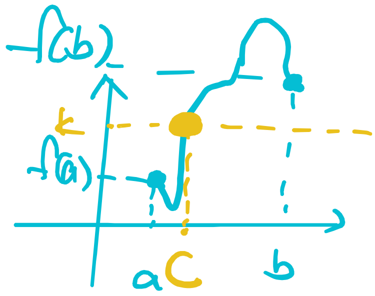
$$= |2| = 2 \quad \#$$

<u>Application to solving equations:</u>

<u>Thm</u> (Intermediate Value Thm, 中間值定理
Thm 2.6.1)

Let f be a continuous function on $[a,b]$. If $k \in \mathbb{R}$ is a number s.t. $f(a) < k < f(b)$

or $f(b) < k < f(a)$,



then $\exists\ c \in (a,b)$ s.t.

$$f(c) = k$$

NOT continuous

$\nexists c$ s.t. $f(c)=k$

pf

Step 1:

Divide $[a,b]$ into $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$

If $f\left(\frac{a+b}{2}\right) = k$, then the conclution follows.

If not, then k is either

between $f(a)$ and $f\left(\frac{a+b}{2}\right)$

or

between $f\left(\frac{a+b}{2}\right)$ and $f(b)$

Let

$$[a_1, b_1] = \begin{cases} \left[a, \frac{a+b}{2}\right] & \text{if } k \text{ is between } f(a) \text{ and } f\left(\frac{a+b}{2}\right) \\ \left[\frac{a+b}{2}, b\right] & \text{if } k \text{ is between } f\left(\frac{a+b}{2}\right) \text{ and } f(b) \end{cases}$$

<u>Step2</u>: Do the same thing for $[a_1, b_1]$:

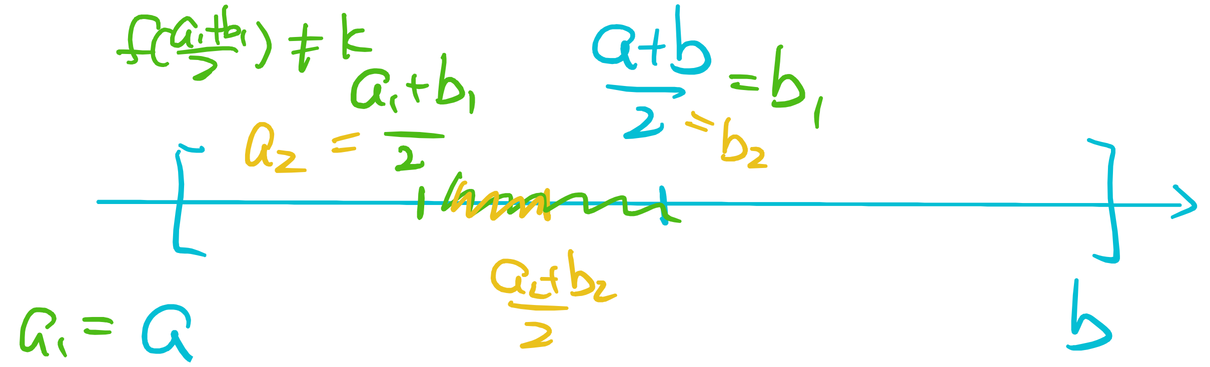
If $f\left(\frac{a_1+b_1}{2}\right) = k$, take $c = \frac{a_1+b_1}{2}$

If not, set

$$[a_2, b_2] = \begin{cases} \left[a_1, \frac{a_1+b_1}{2}\right] & \text{if } k \text{ is between } f(a_1) \text{ and } f\left(\frac{a_1+b_1}{2}\right) \\ \left[\frac{a_1+b_1}{2}, b_1\right] & \text{if } k \text{ is betwee } f\left(\frac{a_1+b_1}{2}\right) \text{ and } f(b_1) \end{cases}$$

$f(a) - k - f\left(\frac{a+b}{2}\right)\ f\left(\frac{a+b}{2}\right) \neq k$

$f\left(\frac{a_1+b_1}{2}\right) \neq k$



Step 3: Repeat this process:

If $f\left(\frac{a_n+b_n}{2}\right) = k$ for some n, then the thm is proved.

If $f\left(\frac{a_n+b_n}{2}\right) \neq k$ $\forall n$, then ....

we obtain a seq of intervals:

$$[a,b] \supseteq [a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \cdots$$

s.t. (i) $b_n - a_n = \frac{b-a}{2^n}$

(ii) k is between $f(a_n)$ and $f(b_n)$

(iii) |

$$a \leq a_1 \leq a_2 \leq a_3 \leq \cdots a_n < b_n \cdots \leq b_3 \leq b_2 \leq b_1 \leq b$$

So by (iii),

$(a_n)_{n=1}^{\infty}$ is increasing and bounded above by b

$(b_n)_{n=1}^{\infty}$ is decreasing and bdd below by a

$\Rightarrow$ $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

By (i),

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\left(\frac{b-a}{2^n}\right) = 0$$

$$\parallel$$

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$

let

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \overset{\downarrow}{=} c \in [a,b]$$

By (ii), k is between $f(a_n)$ and $f(b_n)$

$$\Rightarrow (k - f(a_n))(k - f(b_n)) < 0$$

$$\Rightarrow \overline{\lim_{n\to\infty}}(k - f(a_n))(k - f(b_n)) \leq \underset{\overline{\lim}_{n\to\infty} 0}{\overset{}{0}}$$

$\parallel \leftarrow \lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n \overset{f \text{ conti}}{\Rightarrow} \lim_{n\to\infty} f(a_n) = f(c)$

$$\lim_{n\to\infty}(k-f(a_n)) \cdot \lim_{n\to\infty}(k-f(b_n)) \quad = \lim_{n\to\infty} f(b_n)$$

$$\parallel$$

$$(k-f(c)) \cdot (k-f(c))$$

$$\parallel$$

$$(k-f(c))^2$$

$$\vee\!/$$

$$0$$

$$\left(\begin{array}{l} f(a) \neq k,\ f(b) \neq k \\ \Rightarrow\ c \in (a,b) \end{array} \right)$$

$$\Rightarrow (k-f(c))^2 = 0 \Rightarrow k = f(c)$$

#

<u>Example</u>

① Show that $x^4 - x - 1 = 0$ has a solution in $[-1,1]$.

pf

Let $f(x) = x^4 - x - 1$. which is continuous on $[-1,1]$

Since

$$f(-1) = (-1)^4 - (-1) - 1 = 1 > 0 \quad \text{// } k \text{ in Thm}$$

$$f(1) = 1^4 - 1 - 1 = -1 < 0$$

By IVT (intermediate value thm),

$\exists\, c \in (-1, 1)$ s.t.

$$f(c) = 0$$

That is, c is a sol #

② Show that $x^3-4x+2=0$ has 3 distinct roots in $[-3, 2]$.

has at most 3 roots.

pf

Let $g(x) = x^3-4x+2$ which is Continuous on $[-3,2]$

Since

$$g(-3) = -13 < 0$$
$$g(-2) = 2 > 0$$
$$g(0) = 2 > 0$$
$$g(1) = -1 < 0$$
$$g(2) = 2 > 0$$

$\exists\, c_1 \in (-3, -2)$, $c_2 \in (0, 1)$, $c_3 \in (1, 2)$

s.t. $g(c_1) = g(c_2) = g(c_3)$. #