Calculus, Fall 2025, week 5

Thm (Thm 2.5.1, Pinching thm, sandwich thm)

Let f, g, h be functions defined around $x = c \in \mathbb{R}$. (That is, they are defined on $(c-p, c+p) - \{c\}$, for some $p > 0$.)

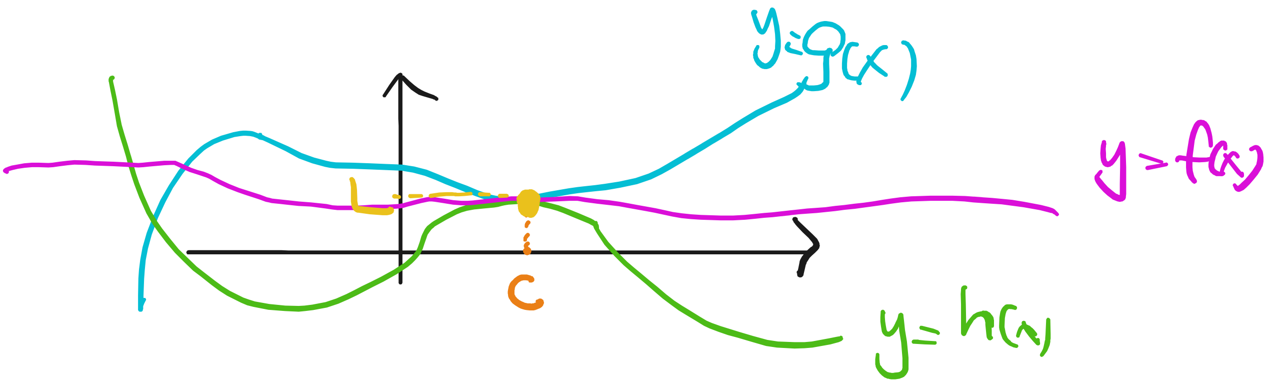
Suppose $\exists \delta > 0$ s.t.

$$h(x) \le f(x) \le g(x)$$

$\forall x \in (c-\delta, c+\delta) - \{c\}$. $\Leftrightarrow$ $0 < |x-c| < \delta$ ①

If $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$, then

$$\lim_{x \to c} f(x) \text{ exists, } = L.$$



Pf. ... Write down the def. of $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$:

Pf: Step 1: Write down

By the assumptions, $\forall \varepsilon > 0$, $\exists \delta_1, \delta_2 > 0$ s.t.

① $|g(x) - L| < \varepsilon \quad \forall \ 0 < |x-c| < \delta_1$

② $|h(x) - L| < \varepsilon \quad \forall \ 0 < |x-c| < \delta_2$

By ①,

$$L - \varepsilon < g(x) < L + \varepsilon \quad \forall \ \underbrace{0 < |x-c| < \delta_1}_{②}$$

$$\geq$$
$$f(x)$$
$$\geq$$

By ②,

$$L - \varepsilon < h(x) < L + \varepsilon \quad \forall \ \underbrace{0 < |x-c| < \delta_2}_{③}$$

Step 2: Use $h(x) \leq f(x) \leq g(x)$

Take

$$\delta_\varepsilon = \min\{\delta_1, \delta_2, \varepsilon\}$$

Then when $\underbrace{0 < |x-c| < \delta_\varepsilon}$, $\Rightarrow$ ①, ②, ③

$$L - \varepsilon \overset{②}{<} h(x) \leq f(x) \leq g(x) \overset{①}{<} L + \varepsilon$$

$$\Rightarrow \quad |f(x) - L| < \varepsilon \quad \forall \ 0 < |x-c| < \delta$$

$\forall\ 0<|x-c|<\delta_\varepsilon$.

So $\lim\limits_{x\to c} f(x) = L$ #

<u>Lemma</u>

① $\lim\limits_{x\to c} f(x) = L \iff$ ② $\lim\limits_{x\to c} |f(x)-L| = 0$

pf

By the definition,

① means:

$\forall \varepsilon>0,\ \exists \delta>0$ s.t.

$|f(x)-L| < \varepsilon \quad \forall\ 0<|x-c|<\delta$

② means:

$\forall \varepsilon>0,\ \exists \delta>0$ s.t.

$\underbrace{\big|\,|f(x)-L|-0\,\big|}_{=\,|f(x)-L|} < \varepsilon \quad \forall\ 0<|x-c|<\delta$

So ① and ② mean exactly the same thing. #

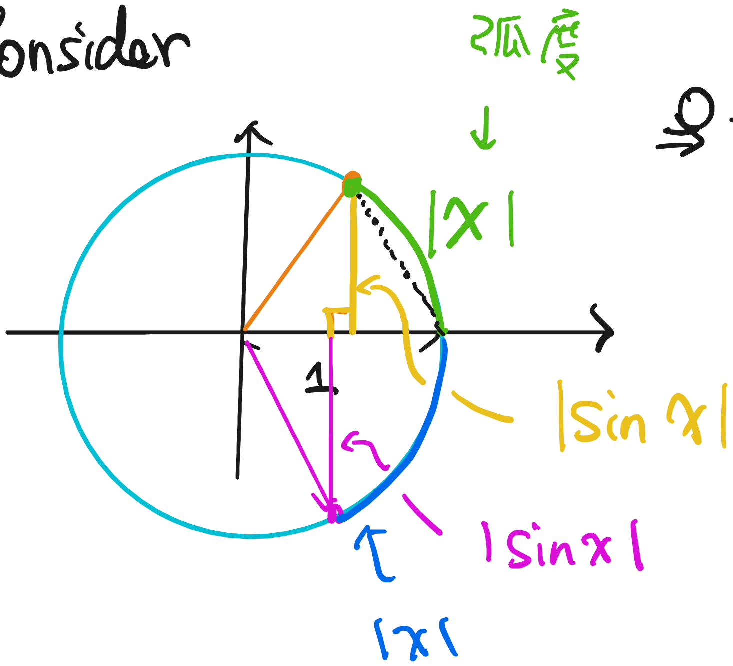
Thm (Equation (2.5.2), (2.5.3))

(i) $\lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = 1$

pf

(i) Consider



$\Rightarrow 0 \le |\sin x| \le |x|$

$\forall \quad |x| \le \frac{\pi}{2}$

Since
$$\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x|$$

by the pinching thm,
$$\lim \overset{= |\sin x - \overset{L}{\overset{\|}{0}}|}{|\sin x|} = 0$$

$x \to 0$

Lemma
$\Rightarrow$

$$\lim_{x\to 0} \sin x = 0 \quad \Rightarrow \text{(i)}$$

Step1:

(ii) For $|x| \le \frac{\pi}{2}$,

$$\cos x = \sqrt{1 - \sin^2 x}$$

Recall that

$$\lim_{x\to 1} \sqrt{x} = \sqrt{1} = 1$$

i.e. $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t.

$$\left|\sqrt{x} - 1\right| < \varepsilon \quad \forall \ 0 < |\underbrace{x}_{(1-\sin^2 x)} - 1| < \delta_\varepsilon$$

Step2:

By (i), $\lim_{x\to 0} \sin x = 0$, so we have

$$\lim_{x\to 0} (1 - \sin^2 x) = \lim_{x\to 0} 1 - \left(\lim_{x\to 0} \sin x\right)^2$$

$$= 1 - 0^2 = 1$$

Step3.

For $\delta_\varepsilon > 0$, $\exists\ \tilde{\delta} > 0$ s.t.

$$\alpha \ |\sin^2 x| = |(1-\sin^2 x) - 1| < \delta_\varepsilon$$

$$\forall\ 0 \le |x - 0| < \tilde{\delta}$$

Step 1

So for $0 < |x-0| < \tilde{\delta}$,

$$\left| \sqrt{1-\sin^2 x} - 1 \right| < \varepsilon$$

So

$$\lim_{x\to 0} \sqrt{1-\sin^2 x} = \lim_{x\to 0} \cos x = 1$$

let $y = 1-\sin^2 x$

as $x \to 0$, $1-\sin^2 x \to 1 \Leftrightarrow y \to 1$

$$\Rightarrow \sqrt{y} \to 1$$

#

Cor

(iii) $\lim_{x\to c} \sin x = \sin c$

(iv) $\lim_{x\to c} \cos x = \cos c$

Pf

(iii) **Note:**

exercise

Recall

$\sin h \cos c + \cos h \sin c$

$$\lim_{x\to c} \sin x = \lim_{h\to 0} \underbrace{\sin(h+c)}$$

constant

$$= \lim_{h\to 0}\left(\sin h \cdot \cos c + \cosh \cdot \sin c\right)$$

$$= \cos c \cdot \underbrace{\left(\lim_{h\to 0} \sin h\right)}_{=0} + \sin c \cdot \underbrace{\left(\lim_{h\to 0} \cos h\right)}_{=1}$$

$$= \sin c \,. \quad \#$$

(iv) $$\lim_{x\to c} \cos x = \lim_{h\to 0} \underbrace{\cos(h+c)}_{=}$$

$$= \lim_{h\to 0}\left(\cos h \cos c - \sin h \sin c\right)$$

$$= \cos c \cdot \underbrace{\lim_{h\to 0} \cos h}_{=1} - \sin c \cdot \underbrace{\lim_{h\to 0} \sin h}_{=0}$$

$= \cos c$ #

Thm (Eq. (2.5.5))

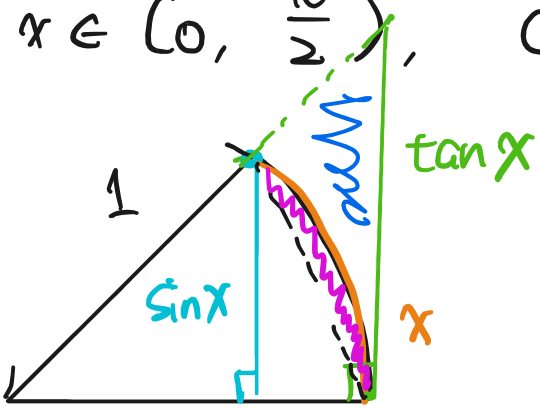
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

pf

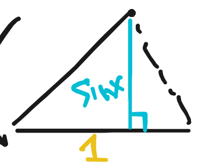
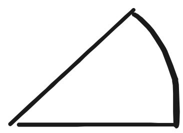
Here we show $\boxed{\lim_{x \to 0^+} \frac{\sin x}{x} = 1 .}$

The eq. $\lim_{x \to 0^-} \frac{\sin x}{x} = 1$ can be proved similarly.

For $x \in (0, \frac{\pi}{2})$, consider

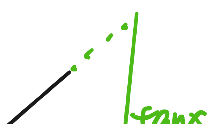


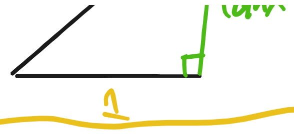
$\Rightarrow$

area() $<$ area() $= \pi \cdot \frac{x}{2\pi} = \frac{x}{2}$

$\|$

$\frac{1}{2} \cdot 1 \cdot \sin x$

$\leq$ area()

$\leq$ area ()

$$= \frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \cdot \frac{\sin x}{\cos x}$$

Note that

$$\frac{\sin x}{2} \leq \frac{x}{2} \Rightarrow \frac{\sin x}{x} \leq 1$$

$$\frac{x}{2} \leq \frac{1}{2} \frac{\sin x}{\cos x} \Rightarrow \cos x \leq \frac{\sin x}{x}$$

So
$$\cos x \leq \frac{\sin x}{x} \leq 1$$

Since
$$\lim_{x \to 0} \cos x = 1 = \lim_{x \to 0} 1$$

by the pinching thm,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad . \#$$

Cor (Eq. (2.5.5))

$$\lim \quad \frac{1 - \cos x}{\quad} = 0$$

$$\lim_{x\to 0} \frac{\quad}{x}$$

$1^2 - \cos^2 x = \sin^2 x$

pf

Since

$$\frac{1-\cos x}{x} = \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$\frac{\sin x}{x} \to 1$, $\sin x \to 0$ as $x \to 0$, $1+\cos x \to 1+1=2$

we have

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x}$$

$\lim_{x\to 0} \frac{\sin x}{x} \to 1$, $\sin x \to 0$, $1+\cos x \to 2$

$$= 1 \cdot \frac{0}{2} = 0 \quad \#$$

Example

① $$\lim_{x\to 0} \frac{\sin(4x)}{3x} = \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{4}{3}$$

$y \to 0 \Leftarrow x \to 0$

let $y = 4x$

$$= \lim_{y\to 0} \frac{\sin y}{y} \cdot \frac{4}{3} = 1 \cdot \frac{4}{3} = \frac{4}{3} \quad \#$$

② $\lim_{x\to 0} x \cdot \cot(3x) = \lim_{x\to 0} x \cdot \frac{\cos(3x)}{\sin(3x)}$

$$= \lim_{x\to 0} \frac{\cos 3x}{\frac{\sin 3x}{3x} \cdot 3}$$

$$= \frac{1}{3} \cdot \frac{\lim_{x\to 0} \cos 3x}{\lim_{x\to 0} \frac{\sin 3x}{3x}}$$

Let $y = 3x$

" $x \to 0$ $\Leftrightarrow$ $y \to 0$ "

$\boxed{\text{exercise}}$

$$= \frac{1}{3} \cdot \frac{\lim_{y\to 0} \cos y}{\lim_{y\to 0} \frac{\sin y}{y}}$$

$$= \frac{1}{3} \quad \frac{1}{1} = \frac{1}{3}.$$

③ $\lim_{x\to \frac{\pi}{4}} \frac{\sin(x - \frac{\pi}{4})}{(x - \frac{\pi}{4})^2}$ does NOT exist, since

let $x - \frac{\pi}{4}$

$\frac{\sin y}{y} \to 1 \neq 0$

$$\lim_{y\to 0} \frac{\sin y}{y^2} = \lim_{y\to 0} \frac{\sin y}{y}\cdot\frac{1}{y} \quad y \to 0$$

$\lim \frac{1}{y}$ case 2

#

Remark

In the computation, we changed the variable

$$\text{“}\ y = x - \frac{\pi}{4}\ \text{”}$$

To be precise, we use the following lemma

Lemma

Suppose that $\qquad y = g(x)$

$$\lim_{x\to c} g(x) = d, \qquad \lim_{y\to d} f(y) = L.$$

If $\exists\ p>0$ s.t. $g(x) \neq d \quad \forall\ 0<|x-c|<p$, then

$$\lim_{x\to c} f(g(x)) = L$$

pf: exercise.
See the lecture notes

For example, in ①,

$$\lim_{x\to 0} \frac{\sin(4x)}{3x} = \frac{4}{3} \lim_{x\to 0} \frac{\sin(4x)}{4x}$$

take $\begin{cases} f(y) = \frac{\sin y}{y} \end{cases} \Rightarrow f(g(x)) = \frac{\sin(4x)}{4x}$

$\left(g(x) = 4x = y \quad (d=0,\ c=0,\ L=1) \right)$

by Lemma

$$= \frac{4}{3} \lim_{y \to 0} \underbrace{f(y)}_{= \frac{\sin y}{y}} = \frac{4}{3}.$$

Limits involving ∞

Def

We say

(i) $\lim_{x \to +\infty} f(x) = L$ if $\forall \varepsilon > 0, \exists M$ s.t.

$(x \to -\infty)$ $\quad |f(x) - L| < \varepsilon \quad \forall\ x > M.$

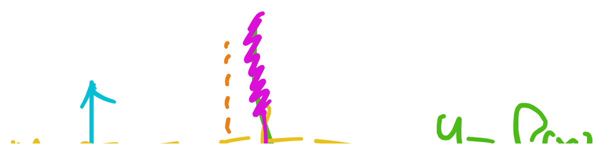
$(\forall\ x < M)$

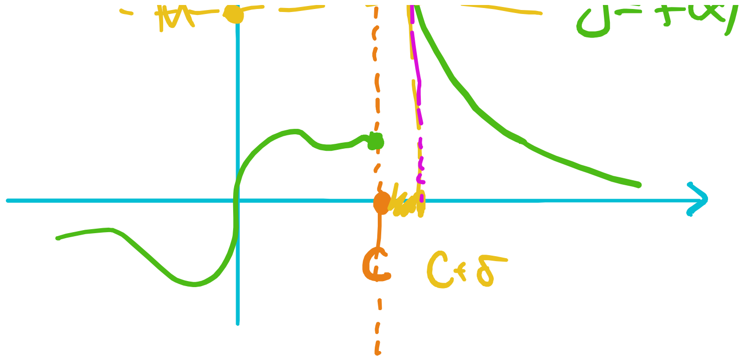
(ii) $\lim_{x \to -\infty} f(x) = L$ if $\forall \varepsilon > 0 \ \exists N$ s.t.

$|f(x) - L| < \varepsilon \quad \forall\ x < N.$

(iii) $\lim_{x \to c^+} f(x) = +\infty$ if $\forall M > 0 \ \exists \delta > 0$ s.t.

$f(x) > M \quad \forall\ c < x < c + \delta$





$$\lim_{x\to c^+} f(x) = -\infty \ , \quad \lim_{x\to c^-} f(x) = +\infty, \quad \lim_{x\to c^-} f(x) = -\infty$$

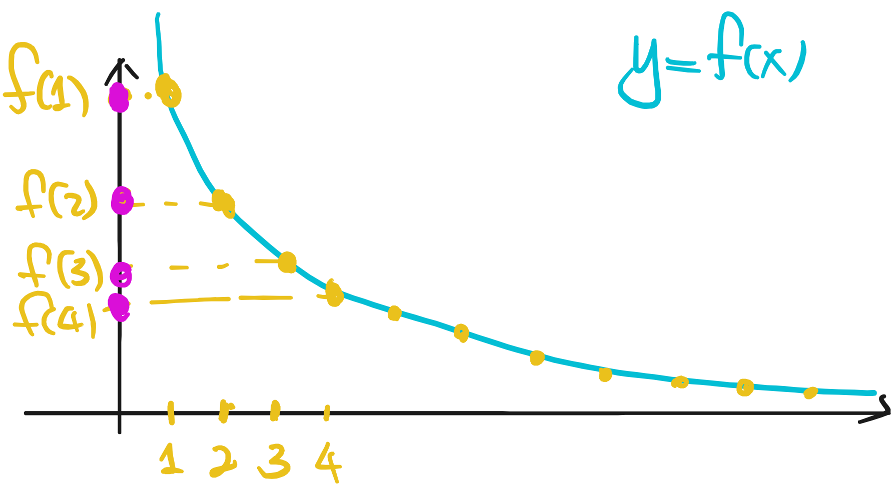
can be defined similarly.

Remark

If $\lim\limits_{\substack{x\to +\infty \\ x\in\mathbb{R}}} f(x) = L$, then the seq.

$$\left(a_n = f(n)\right)_{n=1}^{\infty} = f(1), \ f(2), \ \dots$$

converges to L $\leftarrow$ $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}} f(n) = L$



Example

① $$\lim_{x\to+\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$$

pf

$\forall\, \varepsilon>0,\ \exists\, M=\frac{1}{\varepsilon}>0$ s.t.

$$\left|\frac{1}{x}-0\right| = \frac{1}{|x|} = \frac{1}{x} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

$\forall\ x > M=\frac{1}{\varepsilon}$

② $$\lim_{x\to+\infty} \frac{1}{x^2} = \lim_{x\to+\infty} \frac{1}{x}\cdot\frac{1}{x}$$

$$= \left(\lim_{x\to+\infty} \frac{1}{x}\right)\cdot\left(\lim_{x\to+\infty} \frac{1}{x}\right) \overset{①}{=} 0\cdot 0 = 0$$

③ $$\lim_{x\to+\infty} \frac{(x^2+x)\,\frac{1}{x^2}}{(2x^2+1)\,\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{2+\frac{1}{x^2}}$$

$\to 1+0=1$

$\to 2+0 = 2 \neq 0$

$$= \frac{1}{2}$$

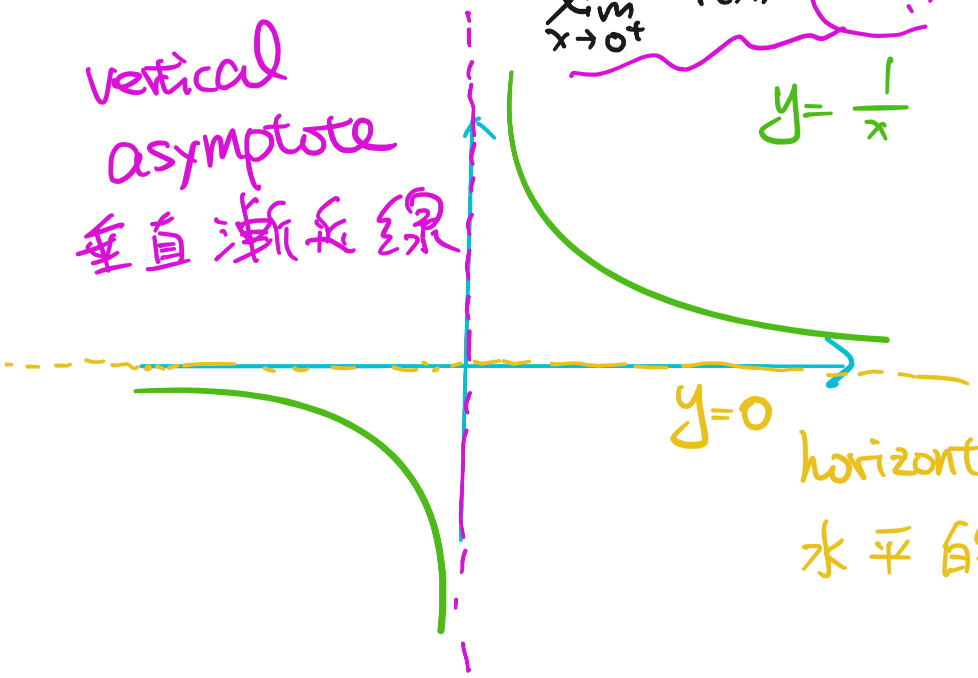
#

Remark

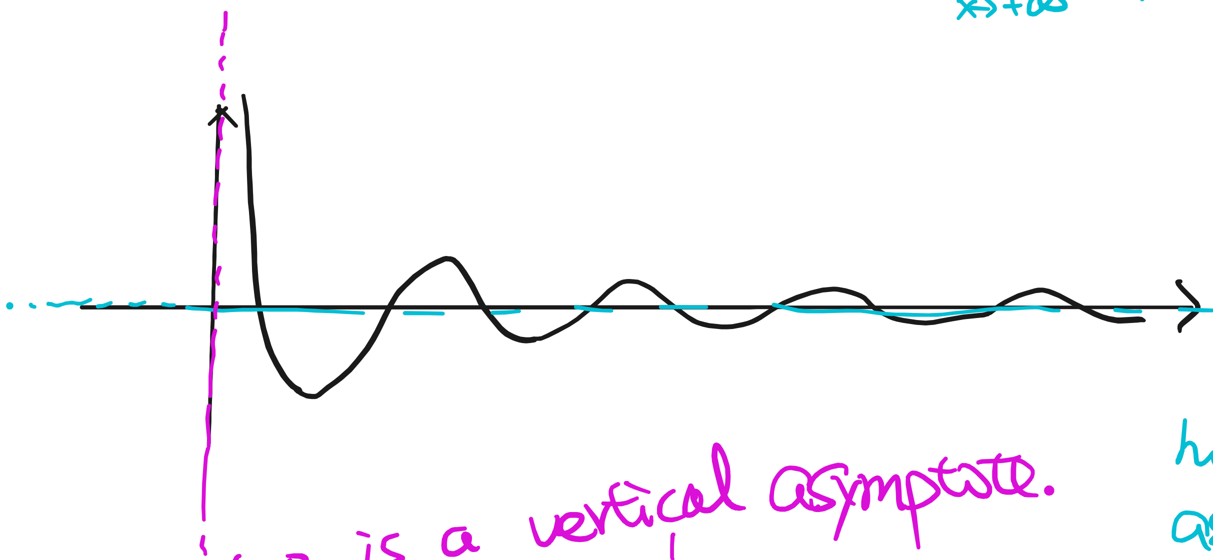
① $f(x)=\frac{1}{x}$: $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

$x=0$

$\lim_{x\to0^+} f(x) = +\infty$, $\lim_{x\to0^-} f(x) = -\infty$



② $f(x) = \dfrac{\cos x}{x}$, $x>0$ $\quad \lim_{x\to+\infty} \dfrac{\cos x}{x} = 0$



$$\lim_{x\to0^+} \frac{\cos x}{\underbrace{x}_{>0}} = +\infty$$

Continuity 連續性

Def (Def 2.4.1 p. 45)

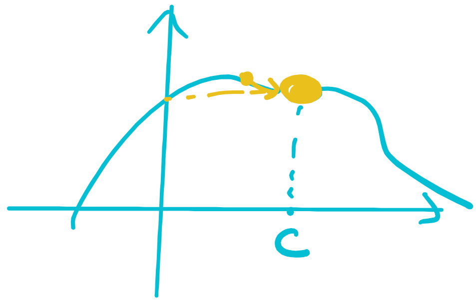
Def (Def. 2.1.1, ...)

Let f be a function defined in

$(c-p,\ c+p)$

for some p. We say f is 連續 **continuous at c** if

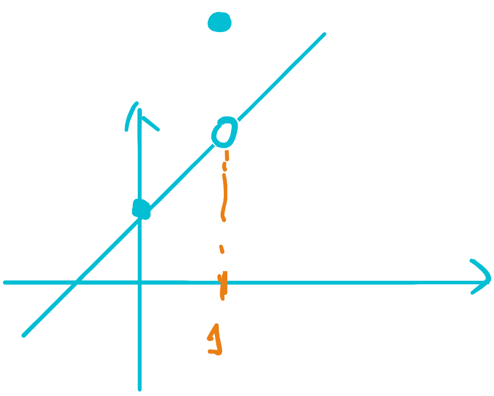
$$\lim_{x \to c} f(x) = f(c)$$



e.g. $f(x) = \begin{cases} x+1 & x \neq 1 \\ 3 & x = 1 \end{cases}$ is NOT continuous at 1

since

$$f(1) = 3$$
$$\neq$$
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x+1$$
$$= 1+1 = 2$$



We say f is **discontinuous at c** if

f is NOT continuous at c.

f is NOT continuous at c.

f is right-continuous at c if

f is defined on $[c, c+p)$ for some $p>0$

$$\lim_{x \to c^+} f(x) = f(c).$$

f is left-continuous at c if

f is defined on $[c-p, c]$ for some $p>0$

$$\lim_{x \to c^-} f(x) = f(c)$$

open interval

f is continuous on $\overbrace{(a, b)}$ if

f is continuous at c for any $c \in (a,b)$

closed interval

f is continuous on $[a, b]$ if

(i) f is continuous on (a, b)

(ii) f is right-continuous at a

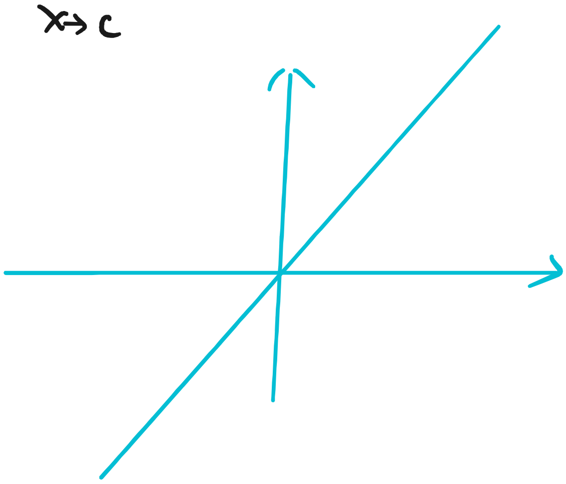
(iii) f is left-continuous at b

Continuous on $(a, b]$ or $[a, b)$ can be defined similarly

Example

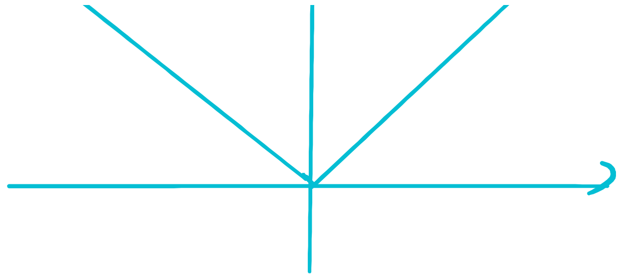
① $f(x) = x$ is continuous on $(-\infty, \infty)$

since $\lim_{x \to c} x = c = f(c)$



② $f(x) = |x|$ is continuous on $(-\infty, \infty)$

since $\lim_{x \to c} |x| = |c| = f(c)$



③ $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

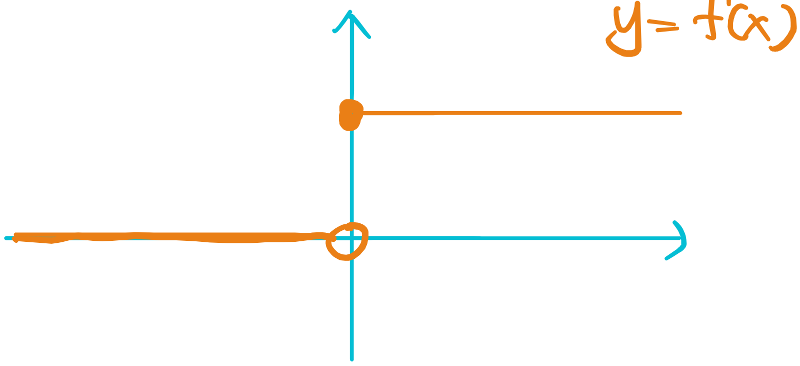
is distinuous at $x=0$. (since $\lim_{x\to 0} f(x)$ does NOT exist)

(iii) fails)

Continuous on $(-\infty, 0)$

Continuous on $[0, +\infty)$

NOT Continuous on $(-\infty, \infty)$



Remark

The following functions are Continuous on $(-\infty, \infty)$:

① any polynomials

② $\sin x$

③ $\cos x$

④ $|x|$

Also,

$$f(x) = \sqrt[n]{x}$$

$\lim_{x\to 0^+} \sqrt[n]{x} = 0$

is continuous on $(0, +\infty)$

(In fact, it is continuous on $[0, +\infty)$)

Remark

A function f is continuous at c

$\Leftrightarrow$ (i) f is defined on $(c-p, c+p)$ for some p

$(\Rightarrow)$ $f(c)$ exists

$\Rightarrow$ e.g. $\frac{1}{x}$ is discontinuous at $x=0$)

(ii) $\lim_{x\to c} f(x)$ exists

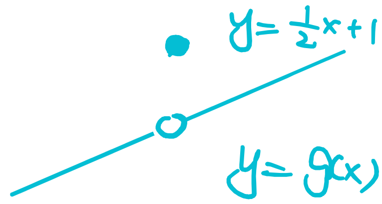
(iii) $\lim_{x\to c} f(x) = f(c)$.

$\Longleftrightarrow$ for any seq. $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = c$,
we have
$$\lim_{n\to\infty} f(x_n) = f(c)$$

<u>Example</u>

① $f(x) = \frac{1}{2}x + 1$ is continuous on $(-\infty, \infty)$

② $g(x) = \begin{cases} \frac{1}{2}x + 1 & x \neq 1 \\ 3 & x = 1 \end{cases}$



is discontinuous at 1, since

$$\lim_{x\to 1} g(x) = \frac{1}{2} + 1 = \frac{3}{2} \neq 3 = g(1).$$

((iii) fails)

<u>Thm</u> (Thm 2.4.2)

If f and g are continuous at c,
and $k \in \mathbb{R}$, then

$f+g$, $\quad f-g$,

$k \cdot f, \quad f \cdot g$

are also continuous at c.

Furthermore, if $g(c) \neq 0$, then

$\frac{f}{g}$ $\leftarrow$ $\lim_{x\to c} \frac{f}{g} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)} = \frac{f}{g}(c)$

$g(c) \neq 0$

is continuous at c

Example

$$f(x) = 3|x| + \frac{x^3 - x}{x^2 - 5x + 6}$$

$= 0 \Leftrightarrow x = 2 \text{ or } 3$

$= (x-2)(x-3)$

is continuous at any $x \neq 2, 3$.

By Thm,

$|x|$ is continuous $\Rightarrow$ $3|x|$ continuous

$\begin{cases} x^3 - x & \text{continuous} \\ x^2 - 5x + 6 & \text{continuous}, \\ \neq 0 & \text{when } x \neq 2, 3 \end{cases}$ $\Rightarrow$ $\frac{x^3 - x}{x^2 - 5x + 6}$ continuous when $x \neq 2, 3$

$\Downarrow$

$$3|x| + \frac{x^3 - x}{x^2 - 5x + 6} \text{ is continuous}$$

at any $x \neq 2, 3$ #

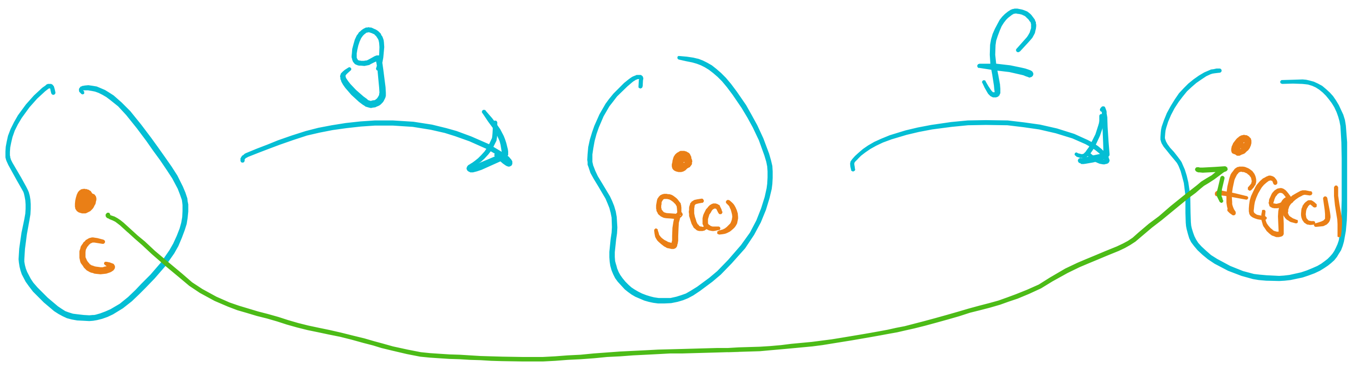
<u>Thm</u> (Thm 2.4.4)

If

(i) g is continuous at c,

(ii) f is continuous at $g(c)$

then $f \circ g$ is continuous at c



pf

We need to show $\lim_{x \to c} f(g(x)) = f(g(c))$:

<u>Step1</u>: Since $\lim_{y \to g(c)} f(y) = f(g(c))$, we have

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $L = f(g(c))$

$$\left| f(y) - f(g(c)) \right| < \varepsilon$$

$y = g(c)$

$\forall\ \cancel{0<}\,|y-g(c)|<\delta$

Note: when $y=g(c)$, $|f(g(c))-f(g(c))|=0<\varepsilon$ is automatically true, in the case of continuous functions

Step2:

For this $\delta>0$, $\exists\ \delta'>0$ s.t.

$$|g(x)-g(c)|<\delta$$

$\forall\ \cancel{0<}\,|x-c|<\delta'$

No need in the case of continuous functions

Step3:

When $(0<)\,|x-c|<\delta'$, we have

$$|\boxed{g(x)}-g(c)|<\delta$$

Step1 $(y=g(x))$ $\Rightarrow$

$$|f(g(x))-f(g(c))|<\varepsilon$$

$\Rightarrow$ $$\lim_{x \to c} f(g(x)) = f(g(c))$$

#

Example

① $f(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous on $(3, \infty)$, since if

$f_1(y) = \sqrt{y}$ ← continuous on $[0, +\infty)$

(*)

$f_2(x) = \dfrac{x^2+1}{x-3}$ ← continuous on $(-\infty, 3) \cup (3, +\infty)$

then $f(x) = (f_1 \circ f_2)(x)$

Note:

$f_2(x) > 0 \quad \forall x > 3$

$f_2(x) < 0 \quad \forall x < 3$

thm (*)

$f(x) = f_1 \circ f_2(x)$ is continuous at x with $x > 3$

② $g(x) = \dfrac{1}{5 - \sqrt{x^2+16}}$ is continuous at any $x \neq \pm 3$

pf: exercise

Assume they are Convergent

1 Problem 1 Prove $\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$

- 大部分同學對極限定義不熟悉或是理解錯誤。
- 重點是給定任意的 ϵ 後要取得一個對應的 N，並說明這個 N 是能滿足要求。
- 由於證明題有嚴格的邏輯順序，在書寫排版時請說明清楚句子的先後順序。太過凌亂，或順序不明的答案有時會使助教難以猜測或誤解你的意思。

2 Problem 3(a) Prove $\overline{\lim}_{n\to\infty}(a_n+b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

- 大部分同學對 $\overline{\lim}$ 的定義不熟或是理解錯誤。
  - 尤其注意 $\overline{\lim}_{n\to\infty} a_n$ 的定義不是 $\lim_{n\to\infty} \underbrace{\sup\{a_n\}_{n=1}^{\infty}}_{= a_n}$。 $\le \sup\{a_n, a_{n+1}, \dots\}$

  $\lim_{n\to\infty}\left(\sup\{a_n, a_{n+1}, a_{n+2}, \dots\}\right)$
- 有一部分同學寫 $a_n \le \overline{\lim}_{n\to\infty} a_n$，這一般來說是不正確的。試考慮 $a_n = \frac{1}{n}$。

$\overline{\lim}_{n\to\infty} \frac{1}{n} = 0$

$a_n = \frac{1}{n} > 0$

$\not\le 0$

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Quartile 0	0
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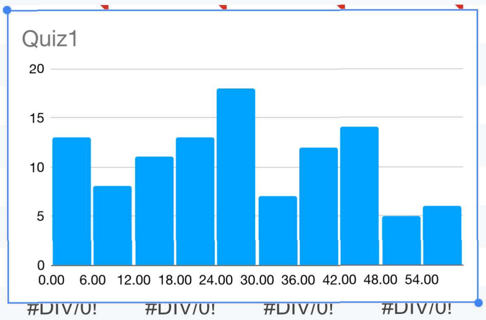

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Incorrect proof (wrong logical order — do not use):

Let $(a_n) \to A$ and $(b_n) \to B$. Fix $\varepsilon > 0$. X Need to consider any $\varepsilon > 0$

Define

$N = \max\{N_1, N_2\}$. =? =? ? X

Since $a_n \to A$, there exists N_1 such that for all $n \ge N_1$, $|a_n - A| < \varepsilon/2$.
Since $b_n \to B$, there exists N_2 such that for all $n \ge N_2$, $|b_n - B| < \varepsilon/2$.
Hence for every $n \ge N = \max\{N_1, N_2\}$ we have both $|a_n - A| < \varepsilon/2$ and $|b_n - B| < \varepsilon/2$. By the triangle inequality,

$$|(a_n + b_n) - (A + B)| \le |a_n - A| + |b_n - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall n \ge N$$

so $\lim_{n\to\infty}(a_n + b_n) = A + B$. ∎

*(Why this is wrong: the step "Define $N = \max\{N_1, N_2\}$." appears **before** N_1 and N_2 are produced — the order is backwards. One cannot legitimately define N using N_1, N_2 before proving those N_1, N_2 exist; the argument is circular or at least misordered.)*