Calculus, Fall 2025, week 4

Recall

$$f'(x) = \lim_{h \to c} \frac{f(x+h) - f(x)}{h}$$

needs limits of functions

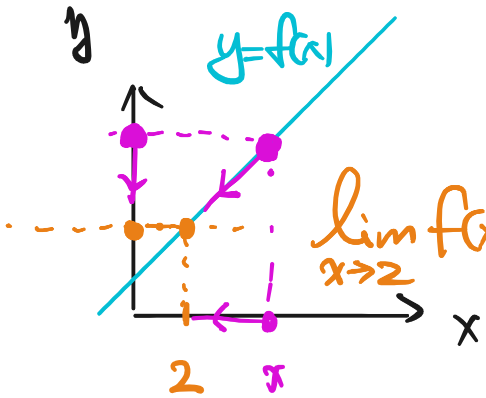
Next topic: $\boxed{\lim_{x \to c} f(x)}$

Intuitive examples

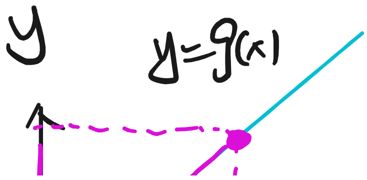
① Let

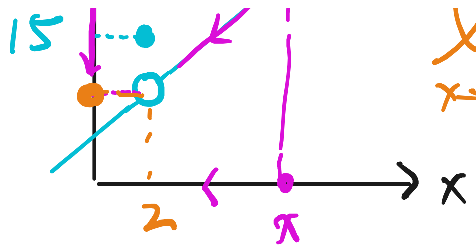
$$f(x) = 4x+5$$

$$g(x) = \begin{cases} 4x+5, & x \neq 2 \\ 15, & x = 2 \end{cases}$$



$$\lim_{x \to 2} f(x) = f(2) = 13$$





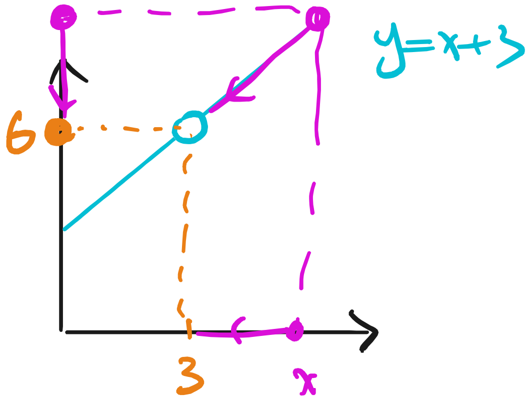
$$\lim_{x\to 2} g(x) = 13 \neq g(2)$$

② Let

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3$$

$$x+3$$

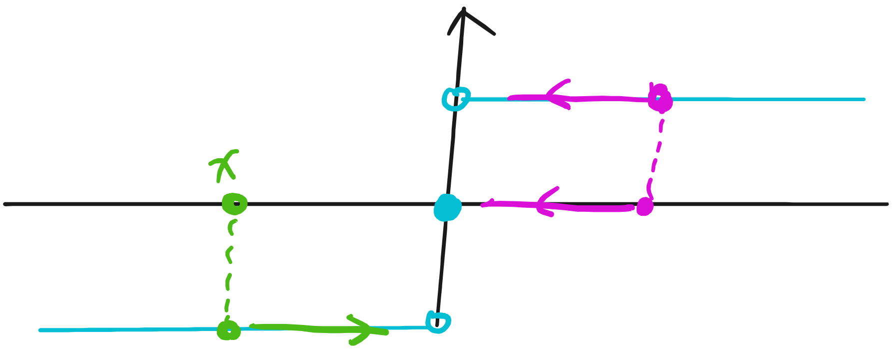
NOT defined at $x=3$



$$\lim_{x\to 3} f(x) = 6$$

③ Let

$$f(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$



$\lim_{x\to 0} f(x)$ does NOT exist

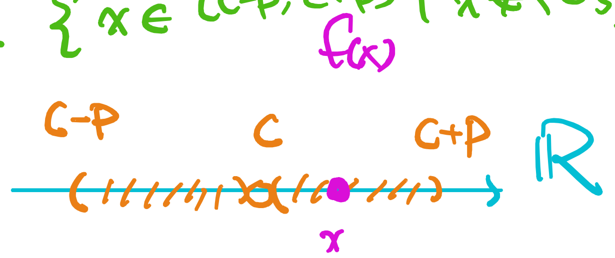
Precise definitions of $\lim_{x\to c} f(x)$ (§2.2)

Def (Def 2.2.1, 2.2.7, 2.2.8)

Let f be a function defined on

$(c-p, c+p) \setminus \{c\}$ $= \{x \in (c-p, c+p) \mid x \notin \{c\}\}$

with $p > 0$.



We say

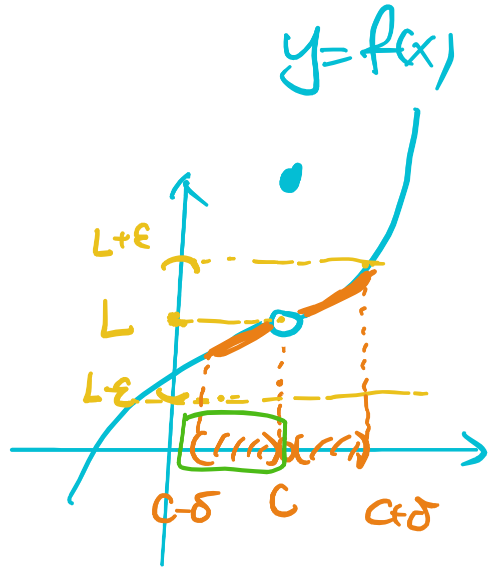
the limit of f as $x \to c$

(i) $\lim_{x\to c} f(x) = L$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$

whenever $0 < |x - c| < \delta$



左極限

(ii) $\lim_{x\to c^-} f(x) = L$ if

$x \to c$

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ s.t.

$|f(x) - L| < \varepsilon$

whenever $\quad c - \delta < x < c$

右極限

(iii) $\lim_{x \to c^+} f(x) = L$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ s.t.

$|f(x) - L| < \varepsilon$

whenever $\quad c < x < c + \delta$

Example

Let

$$g(x) = \begin{cases} 4x + 5, & x \neq 2 \\ 15, & x = 2. \end{cases}$$

Prove $\lim_{x \to 2} g(x) = 13$

pf

$\forall \varepsilon > 0, \quad \exists \delta \overset{\text{"}\delta_\varepsilon\text{"}}{=} \dfrac{\varepsilon}{4} > 0$ s.t.

$$|g(x) - 13| = |(4x+5) - 13|$$

$$= |4x - 8| = 4 \cdot |x-2| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

$\forall\ 0 < |x-2| < \delta$

$\Downarrow$

$x \neq 2$

#

<u>Prop</u>

Let f be a function defined on

$(c-p, c+p) - \{c\}$. Then

(i) $\lim_{x \to c} f(x) = L$ (resp. (ii) $\lim_{x \to c^-} f(x) = L$

(iii) $\lim_{x \to c^+} f(x) = L$)

$\Longleftrightarrow \lim_{n \to \infty} f(x_n) = L$ for any seq.

(resp $x_n \in (c-p, c)$)

$(x_n)_{n=1}^{\infty}$ with $x_n \in (c-p, c+p) - \{c\}$

and $\lim_{n \to \infty} x_n = c$. ($x_n \in (c, c+p)$)

<u>pf of "$\Rightarrow$"</u>

Assume $\lim_{x\to c} f(x) = L$, i.e.

$\forall \varepsilon > 0$, $\exists\, \delta = \delta_\varepsilon > 0$ s.t.

$$|f(x) - L| < \varepsilon \qquad \forall\ 0 < |x - c| < \delta$$

Given any seq. $(x_n)_{n=1}^{\infty}$ with $x_n \in (c-p, c+p) - \{c\}$ $\Downarrow$ $0 < |x_n - c|$

and $\lim_{n\to\infty} x_n = c$, we have

Take $\varepsilon = \delta > 0$

$\cancel{\forall \varepsilon > 0}$ $\exists\, N = \cancel{N_\varepsilon}\ N_\delta$ s.t.

$$|x_n - c| < \cancel{\varepsilon}\ \delta \qquad \forall\ n \geq N_\delta$$

So $\forall\, n \geq N_\delta$, we have

$$0 < |x_n - c| < \delta$$

$$\Rightarrow |f(x_n) - L| < \varepsilon$$

So $\lim_{n\to\infty} f(x_n) = L$ #

Another method:

$\lim_{x\to 0^+} f(x) = 1$

Example

Let $f(x) = \begin{cases} 1 & x > 0 \\ \ldots & x = 0 \end{cases}$

$f(x) = \begin{cases} 0 & x \geq 0 \\ -1 & x < 0 \end{cases}$ $\quad \lim_{x \to 0^-} f(x) = -1$ #

Prove $\lim_{x \to 0} f(x)$ does NOT exist.

pf

Assume $\lim_{x \to 0} f(x)$ exists, $= L$.

Consider

$$\left(x_n = \frac{1}{n}\right)_{n=1}^{\infty} \to 0 \quad \text{as } n \to \infty$$

$\neq 0$

$$\left(y_n = -\frac{1}{n}\right)_{n=1}^{\infty} \to 0$$

$\neq 0$

By Prop,

$$\lim_{n \to \infty} f(x_n) = L$$

$$\|$$

$$\lim_{n \to \infty} f(y_n) = L$$

$\lim_{n \to \infty} f\left(\frac{1}{n}\right) = \lim_{n \to \infty} 1 = 1$

$\#$

$-1 = \lim_{n \to \infty} f\left(\frac{-1}{n}\right) = \lim_{n \to \infty} (-1)$

$(\longrightarrow \longleftarrow)$

So $\lim f(x)$ does NOT exist. #

$\lim_{x\to 0} f(x)$ does ... ... #

Prop (Thm 2.3.1)

Let f be a function defined on $(c-p, c+p) - \{c\}$. Then

(i) $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$

(ii) $\lim_{x\to c} f(x)$ may or may not exist.

If it exists, then it is unique.

Summary

$\lim_{x\to c} f(x)$ exists: we usually certain limit laws to compute it.

$\lim_{x\to c} f(x)$ does NOT exist:

Here are 3 common ways to prove it:

① Prove that $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

② Find a seq $(x_n)_{n=1}^{\infty}$ with $\lim_{n \to \infty} x_n = c$

s.t. $\lim_{n \to \infty} f(x_n)$ does NOT exist.

e.g.

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x)$ does NOT exist, since

for $(x_n = \frac{1}{n})_{n=1}^{\infty}$, we have

$$\lim_{n \to \infty} x_n = 0$$

and

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \frac{1}{\frac{1}{n}} = \lim_{n \to \infty} n$$

does NOT exist

③ Find 2 seq $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ s.t.

$$\lim_{n \to \infty} x_n = c = \lim_{n \to \infty} y_n$$

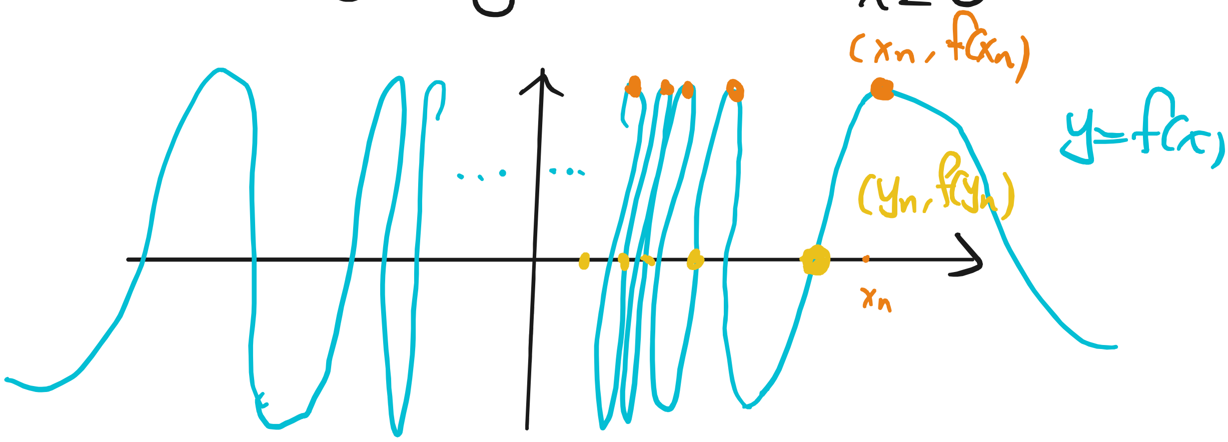
and

and

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

Example

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Prove $\lim_{x\to 0} f(x)$ does NOT exist

pf

Consider

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \neq 0$$

$$y_n = \frac{1}{2n\pi} \neq 0$$

we have

$$\lim x_n = 0 = \lim y_n$$

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$$

and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin \frac{1}{\frac{1}{\frac{\pi}{2}+2n\pi}} = 1$$

$\frac{\pi}{2}+2n\pi$

$$= 1$$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \sin \frac{1}{\frac{1}{2n\pi}} = 0$$

$2n\pi$

$$= 0$$

$\Rightarrow \lim_{x\to 0} f(x)$ does NOT exist. #

Limit laws

We need the following limit laws for computing $\lim_{x\to c} f(x)$

Thm (Thm 2.3.2)

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, and $a, b \in \mathbb{R}$, then we have the following:

(i) $\lim_{x\to c} x = c$

pf

$\forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$ s.t.

$|x - c| < \varepsilon \quad \forall\ 0 < |x - c| < \delta = \varepsilon$ #

(ii) $\lim_{x\to c} |x| = |c|$

$|b-a| \geq |b| - |a|$

$\Uparrow$

pf

$|b-a| \geq |b| - |a|$

$\Rightarrow$ Similarly $|a-b| \geq |a| - |b|$

$\forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$ s.t.

triangle ineq: $|a| + |b-a| \geq |a + b - a| = |b|$

$||x| - |c|| \leq |x - c| < \varepsilon \quad \forall\ 0 < |x - c| < \delta = \varepsilon$

(iii) $\lim_{x\to c} a = a$

pf

$\forall \varepsilon > 0$, $\exists \delta = 1 > 0$ s.t.

$|a - a| = 0 < \varepsilon \quad \forall\ 0 < |x - c| < \delta = 1$ #

(iv)

$$\lim_{x\to c}(f(x)+g(x)) = \underbrace{\lim_{x\to c} f(x)}_{} + \underbrace{\lim_{x\to c} g(x)}_{}$$

$$\lim_{x\to c} f(x) = \alpha, \quad \lim_{x\to c} g(x) = \beta$$

pf

$\forall \varepsilon>0$, $\exists\ \delta_1, \delta_2 > 0$ s.t.

- $|f(x)-\alpha| < \varepsilon/2 \quad \forall\ 0<|x-c|<\delta_1$,
- $|g(x)-\beta| < \varepsilon/2 \quad \forall\ 0<|x-c|<\delta_2$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$

$$\Rightarrow |(f(x)+g(x))-(\alpha+\beta)| = |(f(x)-\alpha)+(g(x)-\beta)|$$

$$\leq |f(x)-\alpha| + |g(x)-\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\forall\ 0<|x-c|<\delta \quad \#$$

(v) $$\lim_{x\to c}(a\cdot f(x)) = a\cdot \lim_{x\to c} f(x)$$

(vi) $$\lim_{x\to c}(f(x)\cdot g(x)) = \Big(\underbrace{\lim_{x\to c} f(x)}_{=\alpha}\Big)\cdot\Big(\underbrace{\lim_{x\to c} g(x)}_{=\beta}\Big)$$

pf

$\forall \varepsilon>0$ $\exists\ \delta_{1,\varepsilon}, \delta_{2,\varepsilon} > 0$ s.t.

$|f(x)-\alpha| < \varepsilon/_{2(|\beta|+1)} \quad \forall\ 0<|x-c|<\delta_{1,\varepsilon/2(|\beta|+1)}$

$|g(x)-\beta| < \dfrac{\varepsilon}{2(|\alpha|+1)} \quad \forall\ 0<|x-c|<\delta_{2,\varepsilon/2(|\alpha|+1)}$

Take $\varepsilon=1$, $\forall\ 0<|x-c|<\delta_{1,1}$

$\Rightarrow |f(x)|-|\alpha| \leq |f(x)-\alpha| < 1 \Rightarrow$ $\underbrace{|f(x)| \leq 1+|\alpha|}$

Take $\delta = \min\{\delta_{1,1},\ \delta_{1,\frac{\varepsilon}{2(|\beta|+1)}},\ \delta_{2,\frac{\varepsilon}{2(|\alpha|+1)}}\}$

$$\Rightarrow \quad |f(x)g(x) - \alpha\beta| = |f(x)g(x) - f(x)\cdot\beta + f(x)\cdot\beta - \alpha\beta|$$

$$\leq \underbrace{|f(x)|}_{\leq} \cdot \underbrace{|g(x)-\beta|}_{<} + |\beta| \cdot \underbrace{|f(x)-\alpha|}_{<}$$

$$< (|\alpha|+1) \cdot \frac{\varepsilon}{2(|\alpha|+1)} + |\beta| \frac{\varepsilon}{2(|\beta|+1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\forall \quad 0 < |x-c| < \delta$ #

Example

① $\lim\limits_{x\to 3} x^2 = \lim\limits_{x\to 3} x \cdot x \overset{(vi)}{=} \lim\limits_{x\to 3} x \cdot \lim\limits_{x\to 3} x$

$\overset{(i)}{=} 3 \cdot 3 = 9.$ #

② $\lim\limits_{x\to 3} (x^2 - x) = \lim\limits_{x\to 3} x^2 + (-1)\cdot x$

$\overset{(iv)}{=} \underbrace{\lim\limits_{x\to 3} x^2}_{\overset{①}{=} 9} + \lim\limits_{x\to 3} (-1)\cdot x$

$\overset{(v)}{=} 9 + (-1) \lim\limits_{x\to 3} x \overset{(i)}{=} 9 + (-1)\cdot 3 = 6$ #

③ $\lim\limits_{x\to -2} |x| (x^3 + 5x) \overset{(vi)}{=} \lim\limits_{x\to -2} |x| \cdot \lim\limits_{x\to -2} (x^3 + 5x)$

$$= 2\left(\underbrace{\lim_{x\to -2} x^3}_{\overset{\text{(vi)}}{=}} + \lim_{x\to -2} 5x\right) \overset{\text{(v)}}{=}$$

$x \cdot x^2$

$|-2| = 2$

$\overset{\text{(iv)}}{=} \lim_{x\to -2} x^3 + \lim_{x\to -2} 5x$

$5 \cdot \lim_{x\to -2} x = 5\cdot(-2) = -10$

$x \cdot x$

$$\lim_{x\to -2} x \cdot \lim_{x\to -2} x^2 = \left(\lim_{x\to -2} x\right)^3 \overset{\text{(i)}}{=} (-2)^3 = -8$$

$$= 2(-8-10) = -36 \ \#$$

<u>Prop</u> (§2.3)

Let $k_1, \cdots, k_n, a_0, \cdots, a_n \in \mathbb{R}$.

Assume $\lim_{x\to c} f(x)$, $\lim_{x\to c} f_i(x)$, $\lim_{x\to c} g(x)$ exist.

Then using (i) – (vi), we have

(vii) $$\lim_{x\to c} (\underbrace{f(x) - g(x)}_{= f(x) + (-1)\cdot g(x)}) \underset{\text{(iv), (v)}}{=} \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

$\lim_{x\to c}$

(viii) $$\lim_{x\to c} \left(k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)\right)$$

$$\overset{}{\underset{(iv),(v)}{=}} k_1 \cdot \lim_{x\to c} f_1(x) + k_2 \cdot \lim_{x\to c} f_2(x) + \cdots + k_n \cdot \lim_{x\to c} f_n(x)$$

(ix) $$\lim_{x\to c} \left(\left(f_1(x) \cdot f_2(x) \cdots \right) \left(f_n(x) \right) \right)$$

$$\underset{(vi)}{=} \lim_{x\to c} f_1(x) \cdot \lim_{x\to c} f_2(x) \cdot \cdots \cdot \lim_{x\to c} f_n(x)$$

(x) $$\lim_{x\to c} \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \right)$$

$$\underset{(viii),(ix)}{=} a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

(xi) Assume $c > 0$, $n \in \mathbb{N}$. Then

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Next: $$\lim_{x\to c} \frac{f(x)}{g(x)} = ?$$

<u>Remark</u> (§2.3)

Assume $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

There 3 cases:

case1 (Th 2.3.8)

Case 1 (Thm 2.5.8)

If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

pf

Step 1: Prove $\lim_{x\to c} \underbrace{\frac{1}{g(x)}} = \dfrac{1}{\lim_{x\to c} g(x)}$

Note: $\frac{1}{g}$ might not be defined everywhere

Assume $\lim_{x\to c} g(x) = \beta \neq 0$

Consider $\varepsilon = \frac{|\beta|}{2} > 0$, $\exists\, \delta_{\frac{|\beta|}{2}} > 0$ s.t.

$$|g(x) - \beta| < \frac{|\beta|}{2} \quad \forall\ 0 < |x-c| < \delta_{\frac{|\beta|}{2}}$$

$$\Leftrightarrow \quad \beta - \frac{|\beta|}{2} < g(x) < \beta + \frac{|\beta|}{2}$$

Note that

- if $\beta > 0$, then $\beta - \frac{|\beta|}{2} = \beta - \frac{\beta}{2} > 0 \Rightarrow g(x) > \frac{\beta}{2} > 0$
- if $\beta < 0$, then $\beta + \frac{|\beta|}{2} = \beta - \frac{\beta}{2} = \frac{\beta}{2} < 0$

$$\Rightarrow g(x) < \frac{\beta}{2} < 0$$

$$\Rightarrow \quad |g(x)| > \frac{|\beta|}{2} > 0 \qquad \forall\ 0 < |x-c| < \delta_{\frac{|\beta|}{2}}$$

Note that

$$\left|\frac{1}{g(x)} - \frac{1}{\beta}\right| = \left|\frac{\beta - g(x)}{\beta \cdot g(x)}\right| = \frac{|g(x)-\beta|}{|\beta|\cdot |g(x)|}$$

$$\frac{|\beta|}{2}$$

$$< \frac{|g(x)-\beta|}{|\beta|\cdot \frac{|\beta|}{2}} \qquad \forall\ 0 < |x-c| < \delta_{\frac{|\beta|}{2}}$$

Since $\lim_{x\to c} g(x) = \beta$, $\forall \varepsilon > 0$, $\exists\, \delta_{\frac{|\beta|^2}{2}} > 0$ s.t.

$$|g(x) - \beta| < \frac{|\beta|^2}{2}\varepsilon \qquad \forall\ 0 < |x-c| < \delta_{\frac{\beta^2}{2}}$$

Take $\delta = \min\left\{ \delta_{\frac{|\beta|}{2}},\ \delta_{\frac{\beta^2}{2}} \right\}$. Then

$$\left|\frac{1}{g(x)} - \frac{1}{\beta}\right| < \frac{|g(x)-\beta|}{\frac{\beta^2}{2}} < \frac{\cancel{\frac{\beta^2}{2}}\cdot\varepsilon}{\cancel{\frac{\beta^2}{2}}} = \varepsilon$$

$$\forall\ 0 < |x-c| < \delta$$

$$\Rightarrow \lim_{x\to c} \frac{1}{g(x)} = \frac{1}{\beta}$$

$\lim_{x\to c} f(x)$

assumption $\frac{1}{\beta}$ as $x \to c$

Step 2

Step 2

by

by Step 1

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\left(f(x)\cdot\frac{1}{g(x)}\right)$$

$$\overset{(vi)}{=} \lim_{x\to c} f(x)\cdot\lim_{x\to c}\frac{1}{g(x)}$$

$$\overset{\text{Step 1}}{=} \lim_{x\to c} f(x)\cdot\frac{1}{\lim_{x\to c} g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \#$$

<u>case 2</u> (Thm 2.3.10)

If $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} f(x) \neq 0$, then

$\lim_{x\to c}\frac{f(x)}{g(x)}$ does NOT exist.

pf

Recall that ① if $\exists\ (x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = c$

and $\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)}$ does NOT exist, then

$\lim_{x\to c}\frac{f(x)}{g(x)}$ does NOT exist.

② If $\lim f(x) = L$ and $\lim x_n = c$ then

② If $\lim_{x\to c} f(x) = L$ and $\lim_{n\to\infty} x_n = c$, then

$$\lim_{n\to\infty} f(x_n) = L$$

Assume $\lim_{x\to c} f(x) = \alpha \neq 0$.

Step 1:

For $\varepsilon = \frac{|\alpha|}{2} > 0$, $\exists\, \delta_{\frac{|\alpha|}{2}} > 0$ s.t.

$$|f(x) - \alpha| < \frac{|\alpha|}{2} \quad \forall\; 0 < |x-c| < \delta_{\frac{|\alpha|}{2}}$$

$$\vee\!/$$

$$|\alpha| - |f(x)|$$

$$\Rightarrow \quad |f(x)| > |\alpha| - \frac{|\alpha|}{2} = \frac{|\alpha|}{2} > 0$$

Take $x_n = c + \frac{1}{n+1}\delta_{\frac{|\alpha|}{2}}$

Since $0 < |x_n - c| = \frac{1}{n+1}\cdot \delta_{\frac{|\alpha|}{2}} < \delta_{\frac{|\alpha|}{2}}$ $\forall n \in \mathbb{N}$.

we have

$$|f(x_n)| > \frac{|\alpha|}{2}$$

Step 2:

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} c + \frac{\delta_{\frac{|\alpha|}{2}}}{n+1} = c$

we have

$$\lim_{n\to\infty} g(x_n) = \lim_{x\to c} g(x) = 0$$

$\Rightarrow$ For $\varepsilon = \frac{|\alpha|}{2n} > 0$ $\quad \exists N_\varepsilon$ $\exists N_{\frac{|\alpha|}{2n}}$ s.t.

$$|g(x_n) - 0| < \varepsilon \qquad \forall n \geq N_\varepsilon$$

$$|g(x_n) - 0| < \frac{|\alpha|}{2n} \qquad \forall n \geq N_{\frac{|\alpha|}{2n}}$$

Choose a subseq. of $(x_n)_{n=1}^{\infty}$:

$$n_1 = \left[N_{\frac{|\alpha|}{2}}\right] + 1 \geq N_{\frac{|\alpha|}{2}} \Rightarrow |g(x_{n_1})| < \frac{|\alpha|}{2}$$

$$n_2 = \max\left\{n_1+1, \left[N_{\frac{|\alpha|}{2\times 2}}\right]+1\right\} \geq N_{\frac{|\alpha|}{2\times 2}} \Rightarrow |g(x_{n_2})| < \frac{|\alpha|}{2\times 2}$$

$$\vdots$$

$$n_{k+1} = \max\left\{n_k+1, \left[N_{\frac{|\alpha|}{2k}}\right]+1\right\} \Rightarrow |g(x_{n_k})| < \frac{|\alpha|}{2k}$$

$$\vdots$$

So we obtain a subseq $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ s.t.

$$|g(x_{n_k})| < \frac{|\alpha|}{2k}$$

Step 3:

Claim: $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}$ does NOT exist

Assume $\left(\frac{f(x_n)}{g(x_n)}\right)_{n=1}^{\infty}$ is Convergent.

Then its subseq. $\left(\frac{f(x_{n_k})}{g(x_{n_k})}\right)_{k=1}^{\infty}$ is also

Convergent. $\Rightarrow$ it is bounded

$\Rightarrow \exists M$ s.t.

$$\left|\frac{f(x_{n_k})}{g(x_{n_k})}\right| \leq M \qquad \forall k \in \mathbb{N}.$$

But

$$M \geq \left|\frac{f(x_{n_k})}{g(x_{n_k})}\right| = \frac{|f(x_{n_k})|}{|g(x_{n_k})|}$$

$|f(x_{n_k})| > \frac{|\alpha|}{2}$ by Step 1

$|g(x_{n_k})| < \frac{|\alpha|}{2k}$ by Step 2

$$> \frac{\cancel{\frac{|\alpha|}{2}}}{\cancel{\frac{|\alpha|}{2}}\frac{1}{k}} = k \qquad \forall k \in \mathbb{N}$$

Take $k = [M] + 1$

$\Rightarrow M > [M]+1 > M$ $(\longrightarrow\!\!\!\!\!\times\longleftarrow)$

So $\left(\frac{f(x_n)}{g(x_n)}\right)_{n=1}^{\infty}$ is divergent

$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)}$ does NOT exist. #

<u>case 3</u>: If $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = 0$, then anything can happen.

Basic technique for <u>case 3</u>:
Simplify $\frac{f(x)}{g(x)}$ before taking the limit.

<u>Example</u>

① $\lim_{x\to 2} \frac{1}{x^3-1} \overset{\text{case 1}}{=} \frac{\lim_{x\to 2} 1}{\lim_{x\to 2}(x^3-1)}$

$(x) \to 2^3-1 = 7 \neq 0$

$= \frac{1}{7}$ #

② $\lim_{x\to 2} \frac{3x-5}{x^2+1} \overset{\text{case 1}}{=} \frac{\lim_{x\to 2} 3x-5}{\lim_{x\to 2} x^2+1}$

$2^2+1 = 5 \neq 0$

$$= \frac{3\cdot 2 - 5}{5} = \frac{1}{5} \quad \#$$

③ $\displaystyle \lim_{x\to 1} \frac{|x|}{x-1}$ (ii) $\to |1| = 1 \neq 0$, $\downarrow$ $1-1=0$

$\underline{\text{case2}}$ does NOT exist.

\#

④ $\displaystyle \lim_{x\to 1} \frac{|x|\,\cancel{(x-1)}}{\cancel{x-1}}$ $\to |1|\cdot(1-1) = 0$, $\downarrow$ $1-1=0$

$$\underset{\text{case3}}{=} \lim_{x\to 1} |x| = |1| = 1 \quad \#$$

⑤ $\displaystyle \lim_{x\to 1} \frac{\cancel{x-1}}{(x-1)^{\cancel{2}}}$ $\to 1-1=0$, $\downarrow$ $(1-1)^2=0$

$$\underset{\text{case3}}{=} \lim_{x\to 1} \frac{1}{x-1}$$

$1 \neq 0$, $x-1 \to 0$ $\leftarrow$ $\underline{\text{case2}}$

does NOT exist. \#

$(\sqrt{x})^2 = x$, $3^2 = 9$, $\to 9-9=0$

$a^2-b^2=(a-b)(a+b)$

⑥ $\displaystyle \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$ $\downarrow (x\to 9)$ $\sqrt{9}-3=0$

$$\underset{\text{case3}}{=} \lim_{x\to 9} \frac{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}{\cancel{\sqrt{x}-3}}$$

$$= \lim (\sqrt{x}+3) = \lim \sqrt{x} + \lim 3$$

$$\lim_{x\to 9} \cdots = \lim_{x\to 9} \cdots + \lim_{x\to 9} \cdots$$

$$= \sqrt{9} + 3 = 6 \quad \#$$

Remark

If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists, then similar to case3, $\lim_{x\to c} \frac{f(x)}{g(x)}$ could be anything.

For example,

divergent

$\times h$

$$\lim_{h\to 0} \frac{1+\frac{1}{h}}{2+\frac{1}{h}} = \lim_{h\to 0} \frac{h+1}{2h+1}$$

$\times h$

divergent

$0+1=1$

$2\cdot 0+1=1\neq 0$

$$= \frac{1}{1} = 1 \quad \#$$

Remark

All the properties in this week hold for one side limits.

hold for one-side limits.

Example

① $\lim_{x\to 0^-} \frac{|x|}{x} \overset{\text{case3}}{=} \lim_{x\to 0^-} \frac{-\cancel{x}}{\cancel{x}}$ -1

$x<0$

$\Rightarrow |x| = -x$

$= \lim_{x\to 0^-} -1 = -1$ #

② $\lim_{x\to 0^+} \frac{x^{\frac{3}{2}}}{|x|} = \lim_{x\to 0^+} \frac{x^{\frac{3}{2}}}{\cancel{x}}$ $= \cancel{x}\cdot\sqrt{x}$

$x>0$

$\Rightarrow |x| = x$

$= \lim_{x\to 0^+} \sqrt{x} = 0$

$\forall \varepsilon>0 \ \exists\, \delta = \varepsilon^2$ s.t.

$|\sqrt{x} - 0| = \sqrt{x} < \sqrt{\varepsilon^2} = \varepsilon$

$\forall\ 0 < x - 0 < \delta = \varepsilon^2$

$\quad\quad\ \ \ \ \, x$

#

Remark

So, "limit calculator" ... we have

Search limit calculator when you have difficulties in computing limits.