Calculus, Fall 2025, week 3

Recall

① A subsequence of a seq $(a_n)_{n=1}^{\infty}$ is a seq of the form

$$a_1, a_2, a_3, \dots$$

strictly increasing

$$(a_{n_k})_{k=1}^{\infty} = a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

e.g. $a_2, a_3, a_5, a_8, \dots$

$\cancel{a_2, a_2}, a_3, \dots$ ← NOT a subseq.

② If $(a_n)_{n=1}^{\infty}$ has two subseq. with different limits, then it is divergent.

e.g. $((-1)^{n+1})_{n=1}^{\infty} = 1, -1, 1, -1, 1, -1, \dots$

conv. to 1

conv. to −1

⇒ it is divergent.

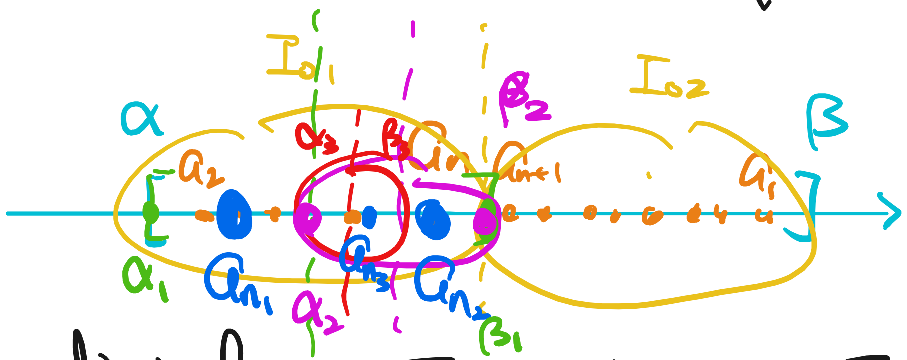
Thm (Bolzano-Weierstrass)

Every bounded seq. has a convergent subsequence.

pf

Let $(a_n)_{n=1}^{\infty}$ be a bounded seq. and let $\alpha, \beta \in \mathbb{R}$ s.t.

$$\alpha \le a_n \le \beta \qquad \forall n \in \mathbb{N}.$$



We divide $I_0 = [\alpha, \beta]$ into two:

$$I_{01} := \left[\alpha, \frac{\alpha+\beta}{2}\right]$$

$$I_{02} := \left[\frac{\alpha+\beta}{2}, \beta\right]$$

Note that at least one of I_{01} or I_{02} contains ∞ terms of $(a_n)_{n=1}^{\infty}$.

Denote this interval by

$$I_1 = [\alpha_1 , \beta_1] \; (= I_{01} \text{ or } I_{02})$$

Again, divide I_1 into two:

$$I_{11} := \left[\alpha_1 , \frac{\alpha_1+\beta_1}{2}\right]$$

$$I_{12} := \left[\frac{\alpha_1+\beta_1}{2} , \beta_1\right]$$

At least one of I_{11} or I_{12} contains ∞ terms of $(a_n)_{n=1}^{\infty}$.

Denote it by I_2 $(= I_{11} \text{ or } I_{12})$

In this way, we obtain

$$I_0 = [\alpha, \beta] \supseteq I_1 = [\alpha_1 , \beta_2] \supseteq I_2 = [\alpha_2 , \beta_2]$$

$$\supseteq \cdots \quad \supseteq I_m = [\alpha_m , \beta_m] \supseteq \cdots$$

s.t.

(i) each I_m contains ∞ terms of $(a_n)_{n=1}^{\infty}$

(ii) $\beta_m - \alpha_m = \dfrac{\beta - \alpha}{2^m} \to 0$ as $m \to \infty$

So we can the following subseq:

Choose

$a_{n_1} \in I_1$

contains ∞ terms of $(a_n)_{n=1}^{\infty}$

$\Rightarrow \exists$ such a_{n_2}

$a_{n_2} \in I_2$, $n_2 > n_1$

$a_{n_3} \in I_3$, $n_3 > n_2$

$\vdots$

$a_{n_m} \in I_m$, $n_m > n_{m-1}$

$I_m = [\alpha_m, \beta_m]$

$\vdots$

Claim: $(a_{n_k})^{\infty}$ is convergent.

Claim $(\alpha_m)_{m=1}$ is convergent

Note that

$$\alpha \le \alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_m$$

$$\cancel{<}\ \beta_m \le \beta_{m-1} \le \cdots \le \beta_2 \le \beta_1 \le \beta$$

$\le$

Since $(\alpha_n)_{n=1}^{\infty}$ is increasing, bounded above by β, it is convergent.

Similarly, since $(\beta_n)_{n=1}^{\infty}$ is decreasing and bounded below by α, it is convergent.

Furthermore, since

$$\beta_m - \alpha_m = \frac{\beta - \alpha}{2^m}$$

we have

$$\lim_{m\to\infty}(\beta_m - \alpha_m) = \lim_{m\to\infty} \frac{\beta - \alpha}{2^m} = 0$$

$\parallel \leftarrow \because \lim_{m\to\infty} \beta_m$ and $\lim_{m\to\infty} \alpha_m$ exist

$$\lim_{m\to\infty} \beta_m - \lim_{m\to\infty} \alpha_m$$

$$\Rightarrow \lim_{m\to\infty} \beta_m = \lim_{m\to\infty} \alpha_m$$

Since $a_{n_m} \in I_m = [\alpha_m, \beta_m]$,

we

$$\alpha_m \le a_{n_m} \le \beta_m \quad \forall m$$

By the pinching thm, $(a_{n_m})_{m=1}^{\infty}$

is convergent and

$$\lim_{m\to\infty} a_{n_m} = \lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m$$

$\Rightarrow$ $(a_{n_m})_{m=1}^{\infty}$ is a convergent subseq. of $(a_n)_{n=1}^{\infty}$ #

Next: Consider the limits of convergent subsequences of a bounded seq. — lim sup, lim inf.

<u>Def</u>

Let S be a bounded subset of $\mathbb{R}$.

最小上界

The least upper bound of S, denoted by $\sup S$, is the number $\alpha \in \mathbb{R}$ with the properties:

(i) $\forall s \in S$, $s \leq \alpha$.

(That is, α is an upper bound of S)

(ii) $\forall \varepsilon > 0$, $\alpha - \varepsilon$ is NOT an upper bound of S anymore, i.e.

$\exists\ S_\varepsilon \in S$ s.t.

$$\alpha - \varepsilon < S_\varepsilon \ (\leq \alpha)$$

e.g. $S = \{\pm 1\} = \{+1, -1\}$

2 is an upper bound of S, but NOT the least upper bound of S.

$1 = \sup S' =$ least upper bound of S

最大下界

Dually, the greatest lower bound of S, denoted by $\inf S$, is the number $\beta = \inf S \in \mathbb{R}$ with the properties:

(i) $\forall s \in S, \quad s \geq \beta$

(ii) $\forall \varepsilon > 0, \quad \beta + \varepsilon$ is NOT a lower bound.

i.e. $\exists\, t_\varepsilon \in S$ s.t.

$$(\beta \leq)\, t_\varepsilon < \beta + \varepsilon$$

Example

① $S = (0,1) \subseteq \mathbb{R}$

$\Rightarrow \sup S = 1, \quad \inf S = 0$

exercise

② $T = \{ 1 + (-\frac{1}{2})^n \mid n = 0, 1, 2, \cdots \}$

$\Rightarrow$ $\sup T = 2$, $\inf T = 1 - \frac{1}{2} = \frac{1}{2}$

Remark

① If $S \subseteq \mathbb{R}$ is nonempty and bounded above (i.e. $\exists M$ s.t. $s \leq M$ $\forall s \in S$)

then $\sup S$ exists in $\mathbb{R}$

This statement is called the Completeness axiom of real numbers.

② If S is nonempty and bounded below (i.e. $\exists m \in \mathbb{R}$ s.t. $s \geq m$ $\forall s \in S$)

then $\inf S$ exists in $\mathbb{R}$.

Let $(a_n)_{n=1}^{\infty}$ be a bounded seq. and let

$$A_n = \{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

Since

$$A_n \supseteq A_{n+1}$$

we have

(i) $\sup A_n \geq \sup A_{n+1}$

(ii) $\inf A_n \leq \inf A_{n+1}$

That is,

(i) $(\sup A_n)_{n=1}^{\infty}$ is decreasing

(ii) $(\inf A_n)_{n=1}^{\infty}$ is increasing.

Since $(a_n)_{n=1}^{\infty}$ is bounded, $\{a_1, a_2, \ldots\} \supseteq \{a_n, a_{n+1}, \ldots\} = A_n$

i.e. $\exists\ \alpha, \beta$ s.t.

$$\alpha \le a_n \le \beta \quad \forall n$$

$\Rightarrow$ α is also a lower bound of $\underline{\underline{A_n}}$

β 〃 upper bound of A_n

$$\Rightarrow \quad \underline{\sup A_n} \ge \alpha \qquad (\sup A_n \ge a_n \ge \alpha)$$

$$\underline{\inf A_n} \le \beta \qquad (\inf A_n \le a_n \le \beta)$$

$\Rightarrow$ $(\sup A_n)_{n=1}^{\infty}$ is decreasing and bounded below

$(\inf A_n)_{n=1}^{\infty}$ is increasing and bounded above

$\Rightarrow$ $\lim_{n\to\infty} \sup A_n$ and $\lim_{n\to\infty} \inf A_n$ exist.

<u>Def</u>

The limit superior of $(a_n)_{n=1}^{\infty}$ is

$$\overline{\lim_{n\to\infty}}\, a_n = \limsup_{n\to\infty} a_n \in \mathbb{R}$$

$$= \lim_{n\to\infty} \sup A_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\}$$

The limit inferior of $(a_n)_{n=1}^{\infty}$

is

$$\underline{\lim}_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \quad \in \mathbb{R}$$

$$= \lim_{n\to\infty} \inf A_n \quad = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \dots\}$$

<u>Prop</u>

Let $(a_n)_{n=1}^{\infty}$ be a bounded seq.

and $\alpha = \overline{\lim}_{n\to\infty} a_n$, $\beta = \underline{\lim}_{n\to\infty} a_n$.

Then $\exists$ subseq $(a_{n_j})_{j=1}^{\infty}$, $(a_{\tilde{n}_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ s.t.

$$\lim_{j\to\infty} a_{n_j} = \alpha$$

$$\lim_{k\to\infty} a_{\tilde{n}_k} = \beta$$

Furthermore, if $(a_{\hat{n}_\ell})_{\ell=1}^{\infty}$ is another arbitrary convergent subseq of $(a_n)_{n=1}^{\infty}$, then

$$\beta \leqslant \lim_{\ell\to\infty} a_{\hat{n}_\ell} \leqslant \alpha$$

Prop

A bounded seq. $(a_n)_{n=1}^{\infty}$ converges

$$\iff \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

Prop

Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$

be bounded (not necessarily convergent) sequences.

Suppose $\exists N \in \mathbb{N}$ s.t.

$$a_n \le b_n \left(\le c_n\right) \quad \forall n \ge N.$$

Then

$$\overline{\lim_{n\to\infty}}\, a_n \le \overline{\lim_{n\to\infty}}\, b_n \left(\le \overline{\lim_{n\to\infty}}\, c_n \right)$$

and

$$\underline{\lim_{n\to\infty}}\, a_n \le \underline{\lim_{n\to\infty}}\, b_n \left(\le \underline{\lim_{n\to\infty}}\, c_n \right)$$

To prove the last proposition, we need a lemma.

<u>Lemma</u>

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent seq. with the property:

$\exists N$ s.t.

$$a_n \leq b_n \quad \forall n \geq N$$

Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

pf

Assume $\alpha := \lim_{n\to\infty} a_n > \lim_{n\to\infty} b_n =: \beta$

For $\varepsilon = \dfrac{\alpha - \beta}{2} > 0$

$\exists$ N_1 and N_2 s.t. $\quad \alpha - \varepsilon < a_n < \alpha + \varepsilon$

$\Leftrightarrow$

$$|a_n - \alpha| < \varepsilon \quad \forall n \geq N_1$$

$$|b_n - \beta| < \varepsilon \quad \forall n \geq N_2$$

$\Leftrightarrow \beta - \varepsilon < b_n < \beta + \varepsilon$

Choose $N_\varepsilon = \max\{N_1, N_2\}$.

$$\Rightarrow \quad a_n > \alpha - \varepsilon = \alpha - \frac{\alpha-\beta}{2} = \frac{\alpha+\beta}{2}$$

$$= \beta + \frac{\alpha-\beta}{2} = \beta + \varepsilon > b_n$$

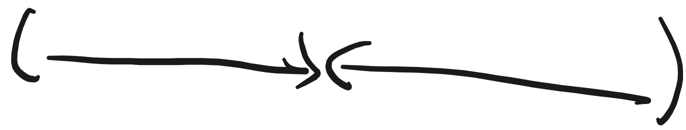
$$\forall\, n \geq N_\varepsilon$$

But the assumption says:

$$a_n \leq b_n \qquad \forall\, n \geq N.$$

For $n_0 = \max\{N_\varepsilon, N\}$,

$$a_{n_0} \leq b_{n_0} < a_{n_0}$$



So

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \qquad \#$$

Prop

Prop

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bdd

s.t. $\exists N$ s.t.

$$a_n \le b_n \quad \forall n \ge N$$

Then

$$\overline{\lim_{n\to\infty}} a_n \le \overline{\lim_{n\to\infty}} b_n$$

$$\underline{\lim_{n\to\infty}} a_n \le \underline{\lim_{n\to\infty}} b_n$$

pf

Let $A_n = \{a_n, a_{n+1}, a_{n+2}, \cdots\}$

$B_n = \{b_n, b_{n+1}, b_{n+2}, \cdots\}$

def
$\Rightarrow$ $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} (\sup A_n)$

$\overline{\lim_{n\to\infty}} b_n = \lim_{n\to\infty} (\sup B_n)$

Since $\sup B_n$ is an upper bound of B_n

we have (inf A_n)

$$\sup B_n \ge b_k \qquad \forall k \ge n$$

For $n \ge N$, (inf $A_n \le a_k \le b_k \quad \forall k \ge n \ge N$)

$$\sup B_n \ge b_k \ge a_k \qquad \forall k \ge n \ge N$$

That is, (inf A_n) $\sup B_n$ is also an (lower) upper bound of A_n $\forall n \ge N$.

$$\Rightarrow \quad \underline{\sup B_n} \ge \underline{\sup A_n} \qquad \forall n \ge N$$

(B_n)

Convergent

(inf $A_n \le$ inf B_n)

By Lemma,

$$\lim_{n\to\infty} \sup B_n \ge \lim_{n\to\infty} \sup A_n$$

$$\overline{\lim_{n\to\infty}}\, b_n = \lim_{n\to\infty} \sup B_n \ge \lim_{n\to\infty} \sup A_n = \overline{\lim_{n\to\infty}}\, a_n$$

The proof for $\underline{\lim_{n\to\infty}}\, b_n \ge \underline{\lim_{n\to\infty}}\, a_n$ is similar. #

Remark

$$\overline{\lim_{n\to\infty}} (a_n + b_n) \leq \overline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$$

Completeness of the real line

Def

A sequence $(a_n)_{n=1}^{\infty}$ is called a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N = N_\varepsilon$

柯西列

s.t.

$$|a_n - a_m| < \varepsilon$$

whenever $n, m \geq N_\varepsilon$.

Prop

Every convergent sequence is a Cauchy sequence.

pf < Let $(a_n)_{n=1}^{\infty}$ be a Convergent seq.

So $\exists$ L s.t.

given $\varepsilon > 0$, $\exists$ $N_{\varepsilon/2}$ s.t.

(*) $|a_n - L| < \frac{\varepsilon}{2}$ $\forall n \geq N_{\varepsilon/2}$

$\Rightarrow \forall n, m \geq N_{\frac{\varepsilon}{2}}$

$$|a_n - a_m| = |(a_n - L) - (a_m - L)|$$

三角不等式

$$\leq |a_n - L| + |a_m - L|$$

$$\overset{(*)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$ $(a_n)_{n=1}^{\infty}$ is Cauchy #

實數的完備性

Thm (Completeness of $\mathbb{R}$)

Every Cauchy seq. (of real numbers) Converges (to a real number).

pf' Outline:

Step 1: Cauchy $\Rightarrow$ bounded $\left(\overset{\text{B-W}}{\Rightarrow} \exists \text{ conv. subseq.} \right)$

Step 2: Cauchy + $\exists$ conv. subseq. $\Rightarrow$ whole seq is convergent

Step 1

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy seq. 2

$\Rightarrow$ For $\varepsilon = 1$, $\exists N_1$ s.t.

$$|a_n - a_m| < 1 \qquad \forall\ n, m \geq N_1$$

$\Rightarrow$ By taking $m = N_1$, we have

$$|a_n - a_{N_1}| < 1 \qquad \forall\ n \geq N_1$$

$\vee\!/$ "Δ ineq."

$$|a_n| - |a_{N_1}|$$

$$\Rightarrow \quad |a_n| < |a_{N_1}| + 1 \qquad \forall\ n \geq N_1$$

Take

$$M = \max\{|a_{N_1}| + 1, |a_1|, |a_2|, \ldots, |a_{N_1}|\}$$

$\ldots, |a_1|, |a_2|, \ldots, |a_{N_1-1}|\}$

$\Rightarrow \quad |a_n| \le M \qquad \forall n \in \mathbb{N}.$

$\Rightarrow \quad (a_n)_{n=1}^{\infty}$ is bdd.

By Bolzano–Weierstrass Thm, $\exists$ convergent subseq $(a_{n_j})_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$

Let $L = \lim_{j\to\infty} a_{n_j}$. ①

Step 2: $\lim_{n\to\infty} a_n = L$.

Given $\varepsilon > 0$, $\exists\, N_1, N_2$ s.t.

① $|a_{n_j} - L| < \varepsilon/2 \quad \forall j \ge N_1$

② $|a_n - a_m| < \varepsilon/2 \quad \forall n, m \ge N_2$

Take $N_\varepsilon = \max\{N_1, N_2\}$.

Note that $n_j \overset{\text{(HW1. 1)}}{\ge} j \ge N_\varepsilon \ge N_2$

$\Rightarrow \forall\ j \geq N_\varepsilon$,

$$|a_j - L| = |a_j - a_{n_j} + a_{n_j} - L|$$

$$\leq \underbrace{|a_j - a_{n_j}|}_{<\,\varepsilon/2\ ②} + \underbrace{|a_{n_j} - L|}_{<\,\varepsilon/2\ ①}$$

$j \geq N_2$, $n_j \geq N_2$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \lim_{j\to\infty} a_j = \lim_{n\to\infty} a_n = L \in \mathbb{R}. \quad \#$$

<u>Example</u>

Let $(a_n)_{n=1}^{\infty}$ be a seq. with the property:

$$|a_{n+1} - a_n| \leq \frac{1}{2^n} \qquad \forall n \in \mathbb{N}$$

Prove that $(a_n)_{n=1}^{\infty}$ is convergent

pf

Note that for $n \geq m$

$$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+2} - a_{m+1}) + (a_{m+1} - a_m)|$$

"$\triangle$ ineq"

$$\leq \underbrace{|a_n - a_{n-1}|}_{\leq \frac{1}{2^{n-1}}} + \underbrace{|a_{n-1} - a_{n-2}|}_{\leq \frac{1}{2^{n-2}}} + \cdots + |a_{m+2} - a_{m+1}| + |a_{m+1} - a_m|$$

By assumption

$$\leq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^{m+1}} + \frac{1}{2^m}$$

$$= \frac{\frac{1}{2^m} - \frac{1}{2^n}}{1 - \frac{1}{2}} \quad \left(= \frac{1}{2^m} \frac{1 - \frac{1}{2^{n-m}}}{1 - \frac{1}{2}} \right)$$

$$\shortparallel$$

$$1 - x^{n-m} = (1-x)(1 + \cdots x^{n-m-1})$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}$$

$$= \frac{1}{2^m} \frac{1 - \frac{1}{2^{n-m}}}{}$$

$$2^m \quad 1-\frac{1}{2} \quad \leq \frac{1}{1-\frac{1}{2}} = 2$$

$$\leq \frac{1}{2^{m-1}}$$

Given $\varepsilon > 0$, $\exists N_\varepsilon = \left(\log_{\frac{1}{2}} \varepsilon\right) + 10$

$$|a_n - a_m| \leq \frac{1}{2^{m-1}} < \left(\frac{1}{2}\right)^{\log_{\frac{1}{2}} \varepsilon} = \varepsilon$$

$\forall\ n \geq m \geq N_\varepsilon$

So $(a_n)_{n=1}^{\infty}$ is Cauchy $\Rightarrow$ $(a_n)_{n=1}^{\infty}$ is convergent #

Relative rate of growth

Let $(a_n)_{n=1}^{\infty}$ be a seq. of positive numbers

$$\left(\Updownarrow \quad \lim_{n\to\infty} \frac{1}{a_n} = 0 \quad \Updownarrow \right)$$

We say

$$a_n \to +\infty$$

as $n \to \infty$ if $\forall M > 0, \exists N = N_M$ s.t.

$$a_n \geq M \qquad \forall n \geq N_M$$

<u>Def</u>

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be seq. of positive numbers.

(i) We say a_n grows faster than b_n

if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0$$

In this case, we also say b_n grows slower than a_n, denoted

by

by

$$b_n = o(a_n)$$

b_n is <u>little-oh</u> of a_n

(ii) We say a_n and b_n <u>grow at the same rate</u> if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L > 0 \qquad L \neq \infty$$

(iii) We write

"b_n is big-oh of a_n"

$$b_n = O(a_n)$$

if $\exists\, M > 0$ s.t.

$$\frac{b_n}{a_n} \leq M \qquad \cancel{\forall n \in \mathbb{N}.}$$

for n sufficiently large

<u>Example</u>

$n = o(n^2)$

① n^2 grows faster than n as $n \to \infty$

since

$$\lim_{n\to\infty} \frac{\quad}{n^2} = \lim_{n\to\infty} \frac{\quad}{n} = 0$$

② Assume $k > l$. $k, l \in \mathbb{N}$.

$$n^l = o(n^k) \leftarrow \lim_{n\to\infty} \frac{n^l}{n^k} = \lim_{n\to\infty} \frac{1}{n^{k-l}} = 0$$

If $k \geq l$, then

$$n^l = O(n^k) \leftarrow \frac{n^l}{n^k} \leq 1 \quad \forall n.$$

③ $2^n = o(n!)$ $\qquad \left(\frac{2^n}{n!} \to 0 \right)$

④ $$2^n = o(3^n) \leftarrow \lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$