Calculus, Fall 2025, week 2

Recall (Thm 11.3.7)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences, and $r \in \mathbb{R}$. Then

(i) $\lim\limits_{n\to\infty}(a_n+b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

(ii) $\lim\limits_{n\to\infty}(r\cdot a_n) = r\cdot \lim\limits_{n\to\infty} a_n$

(iii) $\lim\limits_{n\to\infty}(a_n\cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(iv) if $\lim\limits_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

pf of (iii)

Want to prove $\lim\limits_{n\to\infty}(a_n b_n) = \alpha\beta$

Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$.

Given $\varepsilon > 0$,

$$\exists\ \tilde{N} = \underline{\max\left\{K_1,\ N\left(\frac{\varepsilon/2}{|\beta|+2}\right),\ K\left(\frac{\varepsilon/2}{|\alpha|+1}\right)\right\}}$$

集合中最大的数字

s.t. $\forall\ n \geq \tilde{N}$

$|a_n b_n - \alpha\beta|$ hope $< \varepsilon$

$= |a_n b_n - \alpha b_n + \alpha b_n - \alpha\beta|$

triangle $\leq |a_n b_n - \alpha b_n| + |\alpha b_n - \alpha\beta|$

inequality

$$= |a_n - \alpha|\cdot|b_n| + |\alpha|\cdot|b_n - \beta|$$

① $\wedge|$ hope $\leq$ a fixed number

$$\text{①} \leq \underbrace{\text{②}\ |a_n-\alpha|\cdot(|\beta|+1)}_{\underset{\text{hope}}{<}\ \frac{\varepsilon}{2}} + \underbrace{|\alpha|\cdot \text{③}\ |b_n-\beta|}_{\underset{\text{hope}}{<}\ \frac{\varepsilon}{2}}$$

not always true, need a condition:

Use $\lim_{n\to\infty} b_n = \beta$

For $\varepsilon = 1 > 0$, $\exists\ K_1$ s.t.

$$|b_n - \beta| < 1 \quad \forall\ n \geq K_1$$

$$\vee|$$

$$|b_n| - |\beta|$$

$$\Rightarrow \quad |b_n| < 1 + |\beta| \quad \forall\ n \geq K_1 \quad \text{①}$$

$$\text{②} < \underbrace{\frac{\varepsilon/2}{|\beta|+2}\cdot(|\beta|+1)}_{<\frac{\varepsilon}{2}} + |\alpha|\cdot|b_n - \beta|$$

Use $\lim_{n\to\infty} a_n = \alpha$

need a condition:

For $\dfrac{\varepsilon/2}{|\beta|+2} > 0$, $\exists\ N\left(\dfrac{\varepsilon/2}{|\beta|+2}\right)$ s.t.

$$|a_n - \alpha| < \frac{\varepsilon/2}{|\beta|+2} \quad \forall\ n \geq N\left(\frac{\varepsilon/2}{|\beta|+2}\right) \quad \text{②}$$

$$\text{③} < \frac{\varepsilon}{2} + \underbrace{|\alpha|\cdot\frac{\varepsilon/2}{|\alpha|+1}}_{<\frac{\varepsilon}{2}}$$

need a condition:

For $\dfrac{\varepsilon/2}{|\alpha|+1} > 0$, $\exists\, K\left(\dfrac{\varepsilon/2}{|\alpha|+1}\right)$ s.t.

$$|b_n - \beta| < \frac{\varepsilon/2}{|\alpha|+1} \quad \underbrace{\forall n \geq K\left(\frac{\varepsilon/2}{|\alpha|+1}\right)}_{} \ ③$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\forall \ n \geq \tilde{N} \leftarrow$ choose $\tilde{N}$ s.t.

$n \geq \tilde{N} \Rightarrow$ all ①, ②, ③ are true

\#

Example

"hope to see $\frac{1}{n}$"

① $$\lim_{n\to\infty} \frac{(3n^4 - 2n^2 + 1)/n^5}{(n^5 - 3n^3)/n^5}$$

$$= \lim_{n\to\infty} \frac{3\frac{1}{n} - 2\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^5}{1 - 3\left(\frac{1}{n}\right)^2}$$

NOTE:

① $$\lim_{n\to\infty}\left(3\frac{1}{n} - 2\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^5\right)$$

(i) $0 \ \ \ \ 0 \ \ \ (0)\left(\frac{1}{n}\right)^3 \ \ \ 0 \ \ \ \left(\frac{1}{n}\right)^5$

$$= \lim_{n\to\infty} 3\cdot\frac{1}{n} + \lim_{n\to\infty} (-2)\cdot\left(\frac{1}{n}\right)^3 + \lim_{n\to\infty} \left(\frac{1}{n}\right)^5$$

(ii)

$$= 3\cdot \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0 \text{ last week}} + (-2)\lim_{n\to\infty}\left(\frac{1}{n}\right)^3 + \lim_{n\to\infty}\left(\frac{1}{n}\right)^5$$

(iii)

$$= 3\cdot 0 + (-2)\Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^3 + \Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^5$$

$$= 3\cdot 0 + (-2)\cdot 0^3 + 0^5 = 0$$

② $\lim_{n\to\infty}\left(1-3\left(\frac{1}{n}\right)^2\right)$

(i)

$$= \underbrace{\lim_{n\to\infty} 1}_{=1 \text{ (exercise)}} + \lim_{n\to\infty}(-3)\cdot\left(\frac{1}{n}\right)^2$$

(ii)

$$= 1 + (-3)\lim_{n\to\infty}\left(\frac{1}{n}\right)^2$$

(iii)

$$= 1 + (-3)\Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^2 = 1 \neq 0$$

$$= \frac{\lim_{n\to\infty}\left(3\frac{1}{n} - 2\frac{1}{n^3} + \frac{1}{n^5}\right)}{\lim_{n\to\infty}\left(1-3\frac{1}{n^2}\right)} = \frac{0}{1}$$

$$= 0 \quad \#$$

② $$\lim_{n\to\infty} \frac{(1-4n^7)/n^7}{(n^7+12n)/n^7}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^7} - 4}{1 + \frac{12}{n^6}}$$

NOTE:

$$\lim_{n\to\infty}\left(\frac{1}{n^7} - 4\right) \overset{(ii)}{=} \lim_{n\to\infty}\left(\frac{1}{n}\right)^7 + \underbrace{\lim_{n\to\infty}(-4)}_{=-4}$$

$$\overset{(iii)}{=} \left(\lim_{n\to\infty}\frac{1}{n}\right)^7 - 4 = 0 - 4 = -4$$

$$\lim_{n\to\infty}\left(1 + \frac{12}{n^6}\right) = 1 + 12\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)^6 = 1 \neq 0$$

$$\overset{(iv)}{=} \frac{\lim_{n\to\infty}\left(\frac{1}{n^7} - 4\right)}{\lim_{n\to\infty}\left(1 + \frac{12}{n^6}\right)} = \frac{-4}{1} = -4 \quad \#$$

夾擠定理

Thm ('Pinching thm for seq. Thm 11.5.1)

Suppose that for all n sufficiently large,

$$a_n \le b_n \le c_n.$$

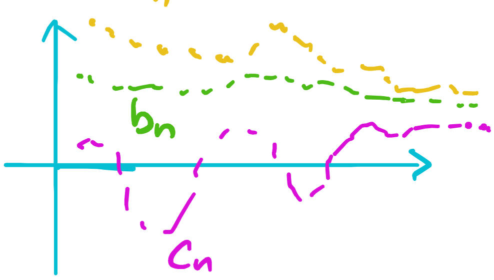
That is, $\exists \tilde{N}$ s.t. $a_n \le b_n \le c_n$ $\forall n \ge \tilde{N}$. condition 1

If

$$\circledast \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n \overset{\text{exist, and}}{=} L$$

then

$$\lim_{n\to\infty} b_n = L.$$



pf.

Given $\varepsilon>0$, (By $\circledast$) $\exists\ \tilde{A}, \tilde{C}$ s.t.

① $|a_n - L| < \varepsilon \qquad \forall n \ge \tilde{A}$

② $|c_n - L| < \varepsilon \qquad \forall n \ge \tilde{C}$

$\Rightarrow$ condition 2

① $L-\varepsilon < a_n < L+\varepsilon \qquad \forall n \ge \tilde{A}$

$L-\varepsilon \quad \cdots \quad L+\varepsilon$

③ $L-\varepsilon < C_n < L+\varepsilon \quad$ condition 3 $\quad \forall n \geq \tilde{C}$

Choose $\tilde{B} = \max\{\tilde{A}, \tilde{C}, \tilde{N}\}$ Then

$$\underbrace{L-\varepsilon < a_n}_{\text{condition 2}} \leq \overbrace{a_n \leq b_n \leq C_n}^{\text{condition 1}} \underbrace{C_n < L+\varepsilon}_{\text{condition 3}}$$

$\Leftrightarrow \quad |b_n - L| < \varepsilon \quad \forall n \geq \tilde{B}$

So $\lim_{n\to\infty} b_n = L$ #

<u>Example</u> Prove the following:

① $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

pf

$-1 \leq \cos n \leq 1$

$\Rightarrow \underset{a_n}{\frac{-1}{n}} \leq \frac{\cos n}{n} \leq \underset{C_n}{\frac{1}{n}}$

Since

$$\lim_{n\to\infty} \frac{-1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

by the pinching thm,

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0 \qquad \#$$

② $$\lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2} = 2$$

pf

Since

$$2 = \sqrt{4} \le \sqrt{4+\left(\frac{1}{n}\right)^2} \le \sqrt{4+2\cdot 2\cdot \frac{1}{n}+\left(\frac{1}{n}\right)^2}$$

$\left(\frac{1}{n}\right)^2 \ge 0$, $2\cdot 2\cdot \frac{1}{n} \ge 0$

$$= \sqrt{\left(2+\frac{1}{n}\right)^2} = 2+\frac{1}{n}$$

$\sqrt{4} \to 2$

$2+\frac{1}{n} \to 2$ as $n\to\infty$

by the pinching thm

$$\lim \sqrt{4+\left(\frac{1}{n}\right)^2} = 2 \qquad \#$$

$n\to\infty$

③ $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

$n! = 1\cdot 2\cdot 3 \cdots n$

$a =$ a fixed real number

pf

$\Updownarrow$ by HW

$\lim_{n\to\infty} \left| \frac{a^n}{n!} - 0 \right| = 0$

$\Updownarrow$

$\lim_{n\to\infty} \frac{|a|^n}{n!} = 0$

It suffices to show $\lim_{n\to\infty} \frac{|a|^n}{n!} = 0$

Note that for $n > |a|$,

$$0 \leq \frac{|a|^n}{n!} = \frac{|a|\cdot|a|\cdot|a| \cdots |a|}{1\cdot 2\cdot 3 \cdots [|a|]} \cdot \frac{|a| \cdots |a|}{([|a|]+1) \cdots n}$$

≤ 1

$\leq \frac{|a|}{n}$

does NOT depend on n

$[|a|]$: the largest integer which $\leq a$

$$\leq \frac{|a|^{[|a|]}}{[|a|]!} \cdot \frac{|a|}{n}$$

$$\to \frac{|a|^{[|a|]}}{\ } \cdot |a| \cdot 0 = 0$$

$[|a|]!$

So by the pinching thm,

$$\lim_{n\to\infty} \frac{|a|^n}{n!} = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{a^n}{n!} = 0 \qquad \#$$

④ Assume $a>0$. Prove that

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

pf

<u>case 1</u>: $a=1 \Rightarrow a^{\frac{1}{n}} = 1 \ \forall n \in \mathbb{N}$.

$$\Rightarrow \lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

$x^n - 1 = (x-1)(1+x+\cdots+x^{n-1})$

NOTE:

$$a-1 = (a^{\frac{1}{n}})^n - 1$$
$$= (a^{\frac{1}{n}}-1)(1+a^{\frac{1}{n}}+(a^{\frac{1}{n}})^2+\cdots+(a^{\frac{1}{n}})^{n-1})$$

<u>case 2</u>: $a>1$.

$$0 \le |a^{\frac{1}{n}} - 1|$$

$$= \frac{|a-1|}{\quad}$$

$$\underbrace{\left|1+\underbrace{a^{\bar{n}}}_{>1}+\underbrace{a^{\bar{\bar{n}}}}_{>1}+\cdots+\underbrace{a^{\bar{n}}}_{>1}\right|}_{> n}$$

$$< \quad \frac{|a-1|}{n} \to 0 \quad n\to\infty$$

So

$$\lim_{n\to\infty} |a^{\frac{1}{n}}-1| = 0$$

by HW

$$\Rightarrow \quad \lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

<u>case 3</u>: $0<a<1$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{a}\right)^{\frac{1}{n}}}$$

$\frac{1}{a} > 1$, by case 2 $\to 1 \neq 0$ as $n\to\infty$

$$= \frac{1}{\lim_{n\to\infty} \left(\frac{1}{a}\right)^{\frac{1}{n}}} = \frac{1}{1} = 1$$

#

Convergence of sequences

Sometimes it is very difficult to compute

$\lim_{n\to\infty} a_n$.

When we don't know $\lim_{n\to\infty} a_n$, we still want to know whether or not $\lim_{n\to\infty} a_n$ exists.

Q: Is $(a_n)_{n=1}^{\infty}$ convergent?

We need some definitions and thms:

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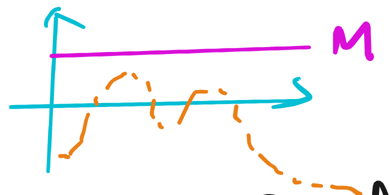
<u>Bounded monotone sequences</u>

We say $(a_n)_{n=1}^{\infty}$ is

e.g.



有上界

<u>bounded above</u> if $\exists M$ s.t. $a_n \le M$ $\forall n \in \mathbb{N}$.

有下界

<u>bounded below</u> if $\exists N$ s.t. $a_n \ge N$ $\forall n \in \mathbb{N}$.

有界

<u>bounded</u> if $\exists M, N$ s.t. $N \le a_n \le M$ $\forall n \in \mathbb{N}$.

<u>unbounded</u> = not bounded

嚴格遞增 (= increasing in textbook)

strictly increasing if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$

strictly increasing if $a_n \lneq a_{n+1}$ $\forall n \in \mathbb{N}$.

遞增 (= nondecreasing in textbook)

increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$ (e.g. $a_n = n^2$)

e.g. 1, 1, 2, 3, 5, 8, . . .

嚴格遞減 (= decreasing in textbook)

strictly decreasing if $a_n \gneq a_{n+1}$ $\forall n \in \mathbb{N}$

遞減

decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$

(= nonincreasing in textbook)

常數的

constant if $a_n = a_{n+1}$ $\forall n \in \mathbb{N}$.

Thm (§11.3)

(i) Every convergent sequence is bounded

$P \Rightarrow Q$

(ii) Every unbounded sequence is divergent

非Q $\Rightarrow$ 非P

(iii) If a sequence is increasing and bounded above, then it is convergent.

(iv) If a sequence is decreasing and

bounded below, then it is convergent.

Example

① $(a_n = n)_{n=1}^{\infty}$ is divergent, since $(n)_{n=1}^{\infty}$ is unbounded.

② $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is convergent, since it is decreasing $\left(\frac{1}{n} > \frac{1}{n+1} \quad \forall n \in \mathbb{N} \right)$ and bounded below by 0 $\left(\frac{1}{n} > 0 \quad \forall n \in \mathbb{N} \right)$

③ $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ $= 1 - \frac{1}{n+1}$ is convergent, since it is increasing $\left(\frac{n}{n+1} < \frac{n+1}{n+2} \quad \forall n \in \mathbb{N} \right)$ and bounded above by 1 $\left(\frac{n}{n+1} < 1 \right)$ #

④ $\left(\frac{n}{2^n}\right)_{n=1}^{\infty}$ is convergent, since it is decreasing:

$$\frac{n+1}{2^{n+1}} - \frac{n}{2^n}{\times 2} = \frac{n+1-2n}{2^{n+1}}$$

$$= \frac{1-n}{2^{n+1}} \le 0 \quad \forall n \ge 1$$

$n \ge 1 \Downarrow$ $1-n \le 0$

and bounded below by 0

$\left(\frac{n}{2^n} > 0 \quad \forall n \ge 1 \right)$

Remark

( ) $\xRightarrow{\text{imply}}$ ( )

① In Thm, (i) $\Longleftrightarrow$ (ii')

② If (iii') is true, then, given any decreasing, bounded below sequence $(b_n)_{n\ge1}$, we consider

$b_{n+1} \le b_n \quad \forall n \ge 1$

$\exists N$ s.t. $b_n \ge N \quad \forall n \ge 1$

定義成

$$a_n := -b_n$$

Recall

$a \le b \Rightarrow -a \ge -b$

Then

$$a_{n+1} = -b_{n+1} \ge -b_n = a_n \quad \text{“increasing”}$$

and

$$a_n = -b_n \le -N \quad \forall n \ge 1 \quad \text{“bounded above”}$$

an upper bound of $(a_n)_{n=1}^{\infty}$
上界

(iii) $\Rightarrow$ $\lim_{n\to\infty} a_n$ exists.

$\Rightarrow$ $\lim_{n\to\infty} (-1)\cdot a_n$ also exists

$$\lim_{n\to\infty} (-1)(-b_n) = \lim_{n\to\infty} b_n$$

" (iii) $\Rightarrow$ (iv) "

That is, $(b_n)_{n=1}^{\infty}$ is convergent. ✗

③ So to prove Thm, it suffices to (只需要)

prove (i) and (iii)

pf of (i)

Assume $(a_n)_{n=1}^{\infty}$ is convergent.

That is, $\exists L \in \mathbb{R}$ with the property:

$\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ s.t.

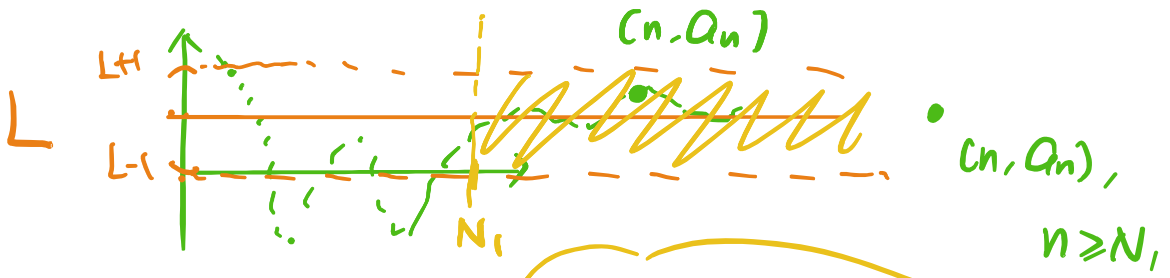
$$|a_n - l| < \varepsilon \qquad \forall n \geq N.$$

Take $\varepsilon = 1 > 0$. $\exists\ N = N_1$ s.t.

$$|a_n - L| < 1 \qquad \forall\, n \geq N_1$$

$$\Rightarrow -1 + L < a_n - L + L < 1 + L$$

$$\Rightarrow L - 1 < a_n < L + 1 \qquad \forall\, n \geq N_1$$



$\underbrace{a_1, a_2, \dots, a_{N_1-1}}_{\text{only finite numbers}}$, $a_{N_1}, a_{N_1+1}, \dots$ $< L+1$, $> L-1$

$\Rightarrow \exists$ the largest one : a_M

$\exists$ the smallest one : a_m

Let

$$a_M = \max\{a_1, a_2, \dots, a_{N_1-1}\}$$

$$a_m = \min\{a_1, a_2, \dots, a_{N_1-1}\}$$

$$\Rightarrow \quad a_m \leqslant a_1, \cdots, a_{N_1-1} \leqslant a_M$$

In summary, we have

- $B \leqslant L-1 < a_n < L+1 \leqslant A \quad \forall n = N_1, N_1+1, N_1+2, \cdots$
- $B \leqslant a_m \leqslant a_n \leqslant a_M \leqslant A \quad \forall n = 1, 2, \cdots, \cancel{N_1}\ N_1-1$

Take

$$A = \max\{L+1,\ a_M\}$$

$$B = \min\{L-1,\ a_m\}$$

$$\Rightarrow \quad B \ \cancel{<}\leqslant\ a_n \ \cancel{<}\leqslant\ A \quad \forall n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad (a_n)_{n=1}^{\infty} \text{ is bounded.} \qquad \#$$

it actully closely relates to the def. of real

To prove (iii),

we a lemma

↑

引理

= 用來輔助證明主定理的定理

numbers

Here, we consider a real number as an infinite decimal:

$$a_0 . a_1 a_2 a_3 a_4 \dots$$

Lemma

If $(a_n)_{n=1}^{\infty}$ is a bounded increasing sequence of integers, then $\exists N$.

s.t.

$$a_n = a_m \qquad \forall n, m \geq N$$

pf

$(a_n)_{n=1}^{\infty}$ is bounded $\Rightarrow \exists M$ s.t.

$$a_n \leq M \quad \forall n = 1, 2 \cdots$$

(反證法)

Assume the conclusion fails.

Then, $\forall N \in \mathbb{N}$, $\exists\ n(N) > N$ s.t.

(a_n) is increasing

$$a_{n(N)} \neq a_N \quad \xrightarrow{(a_n) \text{ is increasing}} \quad a_{n(N)} \underset{\neq}{\geq} a_N$$

Since a_n, a_N are integers,

$$a_{n(N)} - a_N \geq 1$$

We write $n^0(N) = N$

$$n^1(N) = n(N)$$

$$n^2(N) = n(n(N))$$

$$\vdots$$

$$n^j(N) = n(n^{j-1}(N)) = \underbrace{n(n(n \dots n}_{j \text{ times}}(N) \dots))$$

Therefore, we have

$$a_{\underset{n(n^{j-1}(N))}{\underset{\shortparallel}{n^j(N)}}} - a_{n^{j-1}(N)} \geq 1 \quad \forall N \in \mathbb{N}$$

$$\Rightarrow \left(\text{Take } N=1, \quad K = \underbrace{[M-a_1]+1}_{> M-a_1}\right)$$

$$a_{n^K(1)} = a_{n^{K-1}(1)} + \underbrace{a_{n^K(1)} - a_{n^{K-1}(1)}}_{\geq 1}$$

$$\geq a_{n^{k-1}(1)} + 1$$

$$\geq a_{n^{k-2}(1)} + 1 + 1$$

$$\geq \cdots$$

$$\geq a_{\underbrace{1}_{= n^0(1)}} + \overbrace{1 + 1 + \cdots 1}^{K}$$

$$= a_1 + K > a_1 + M - a_1$$

$$= M \geq a_{n^k(1)} \; (\rightarrow\leftarrow)$$

So $\exists N$ s.t. $a_n = a_m \;\; \forall n, m \geq N$ #

<u>pf of (iii)</u>

Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and M an upper bound:

$$a_n \leq M \quad \forall \; n = 1, 2, \cdots$$

Since a_n and M are real numbers, they can be represented by infinite

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decimals : $a_{n0} \in \mathbb{Z} = \{\text{integers}\}$

a_{ni}, $i \geq 1$, $\in \{0, 1, 2, \ldots, 9\}$

$$a_1 = a_{10} . a_{11} \, a_{12} \, a_{13} \cdots$$

$$a_2 = a_{20} . a_{21} \, a_{22} \, a_{23} \cdots$$

$$\vdots$$

$$M = \alpha_0 . \alpha_1 \, \alpha_2 \, \alpha_3 \cdots$$

$\overset{(0)}{\Rightarrow}$ $(a_{n0})_{n=1}^{\infty}$ is an increasing seq. of integers, bounded above by α_0

$\Rightarrow$ By Lemma, $\exists \, N_0$ s.t.

$$a_{n0} = a_{N_0 0} \quad \forall \, n \geq N_0$$

(1) 小數點後第一位:

Consider

$$a_{N_0}, \ a_{N_0+1}, \ a_{N_0+2}, \ \cdots, \ M$$

The seq. $(a_{n1})_{n=N_0}^{\infty}$ is an increasing seq. bounded above 9

Lemma
$\Rightarrow \exists N_1$ s.t.

$$a_{n1} = a_{N_1 1} \quad \forall n \geq N_1$$

(2), (3), $\cdots$:

Similarly, we can find

$$N_0 \leq N_1 \leq N_2 \leq N_3 \leq \cdots.$$

s.t.

$$a_{nk} = a_{N_k k} \quad \forall n \geq N_k.$$

$\forall k = 0, 1, 2, \cdots$

Consider

$$L = a_{N_0 0} \,.\, a_{N_1 1}\, a_{N_2 2}\, a_{N_3 3} \cdots$$

Then one can show that

$$\lim_{n\to\infty} a_n = L \quad \text{(exercise)}$$

#

Subsequences of bounded sequences

Main question: Is a bounded seq convergent?

Def

子數列

A subsequence of a seq. $(a_n)_{n=1}^{\infty}$ is a seq. of the form

$$(a_{n_k})_{k=1}^{\infty} = a_{n_1}, a_{n_2}, a_{n_3}, \cdots$$

where $(n_k)_{k=1}^{\infty}$ is a strictly increasing seq. of positive integers.

<u>Example</u>

Let $(a_n = (-1)^{n+1})_{n=1}^{\infty} = 1, -1, 1, -1, \dots$

Then
$$\begin{matrix} a_1 & a_3 & a_5 & \\ \| & \| & \| & \\ 1, & 1, & 1 & \dots \end{matrix} \quad (n_k = 2k-1)_{k=1}^{\infty} = 1, 3, 5, \dots$$

and
$$\begin{matrix} a_2 & a_4 & a_6 & \\ \| & \| & \| & \\ -1, & -1, & -1, & \dots \end{matrix} \quad (n_k = 2k)_{k=1}^{\infty} = 2, 4, 6, \dots$$

are subsequences of $(a_n)_{n=1}^{\infty}$

<u>Prop</u>

A sequence $(a_n)_{n=1}^{\infty}$ converges to L

$\Leftrightarrow$ = if and only if = iff = 若且唯若

Every subsequence of $(a_n)_{n=1}^{\infty}$ converges to L

Example

Prove $\left((-1)^{n+1} = a_n\right)_{n=1}^{\infty}$ is divergent.

pf (反證法)

Assume $\lim\limits_{n\to\infty} a_n$ exists, $= L$.

By Prop, Take $n_k = 2k-1$, $m_k = 2k$

$$L = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} 1 = 1$$

$1, 1, 1, 1, 1, \ldots$

$$\| \qquad\qquad\qquad\qquad //$$

$$\lim_{k\to\infty} a_{m_k} = \lim_{k\to\infty} (-1) = -1$$

$(\rightarrow\leftarrow)$

So $\lim\limits_{n\to\infty} a_n$ does NOT exist.

i.e. $(a_n)_{n=1}^{\infty}$ is divergent. #

Thm (Bolzono-Weierstrass Thm).

Every bounded sequence has a Convergent subsequence