Calculus, Fall 2025, week 15

Example

Recall

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p>1 \\ \text{diverges}, & p\le 1 \end{cases}$$

① $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = ?$

Sol

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \underbrace{\int_c^1 \frac{1}{x^p}\,dx}$$

$$= \begin{cases} \log x \Big|_c^1 = -\log c, & p=1 \\ \dfrac{x^{-p+1}}{-p+1}\Big|_c^1, & p\ne 1 \end{cases}$$

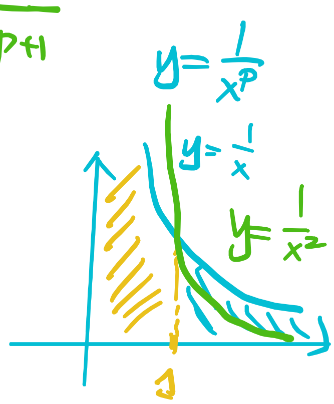
$$= \frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}$$

case1: $p>1$

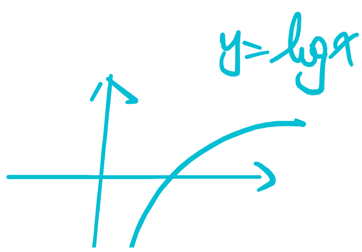
$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+}\left(\frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}\right)$$

$\underbrace{-p+1}_{<0}$, $c^{-p+1} = \dfrac{1}{c^{p-1}}$

diverges (as $c\to 0^+$)



case2: $p=1$



$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} (-\log c) \quad \text{diverges}$$

<u>case 3</u>: $p<1$

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+}\left(\frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}\right)$$

$-p+1 > 0$, $\to 0$

$$= \frac{1}{-p+1}$$

<u>Conclusion</u>

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} = \dfrac{1}{1-p}, & p<1 \\ \text{diverges}, & p\geq 1 \end{cases}$$

#

② $$\int_0^1 \frac{1}{1-x}\,dx = \lim_{c\to 1^-}\left(\int_0^c \frac{1}{1-x}\,dx\right)$$

$$= -\log|1-x|\Big|_0^c$$

$=$

diverges

$$-\log(1-c)$$
$$\downarrow$$
$$\infty \quad \text{as} \quad c \to 1^-$$
$$\#$$

5. $$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$= \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^-} \int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$= \lim_{c\to 1^-} \Big(\underbrace{3(x-1)^{\frac{1}{3}} \Big|_0^c}_{3(c-1)^{\frac{1}{3}} - 3\sqrt[3]{-1} \;=\; 3(c-1)^{\frac{1}{3}} + 3} \Big) = 3$$

$$\int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^+} \int_c^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$= \lim_{c\to 1^+} \Big(\underbrace{3(x-1)^{\frac{1}{3}} \Big|_c^3}_{=} \Big) = 3\sqrt[3]{2}$$

$3\sqrt[3]{2} - 3\cdot(0-1)$

So

$$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = 3 + 3\sqrt[3]{2} \qquad \#$$

<u>Thm</u> (11.7.2)

Let $f, g \in C^0[a,\infty)$, and

$$0 \le f(x) \le g(x) \qquad \forall x \in [a,\infty).$$

Then

(i) $\int_a^\infty g(x)\,dx$ Converges

$\Rightarrow \int_a^\infty f(x)\,dx$ also Converges

(ii) $\int_a^\infty f(x)\,dx$ diverges

$\Rightarrow \int_a^\infty g(x)\,dx$ also diverges.

Pf

Remark

(If $h(x)$ is increasing, and $\exists M$ s.t. $h(x) \le M$ $\forall x \ge a$

then $\lim_{x\to\infty} h(x)$ exists.)

Assume $\int_a^\infty g(x)\,dx$ converges. to M.

Since $0 \le f(x) \le g(x)$ $\forall x \ge a$,

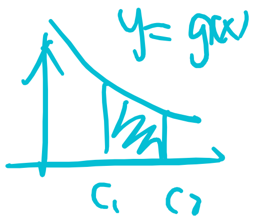
$$h(c) := \int_a^c g(x)\,dx$$

is increasing ← if $c_1 < c_2$, then

$$h(c_2) - h(c_1)$$

$$= \int_a^{c_2} g(x)\,dx - \int_a^{c_1} g(x)\,dx$$

$$= \int_{c_1}^{c_2} g(x)\,dx \ge 0$$



i.e, $h(c_2) \ge h(c_1)$

Similarly,

$$\tilde{h}(c) := \int_a^c f(x)\,dx$$

is also increasing

σ

Furthermore

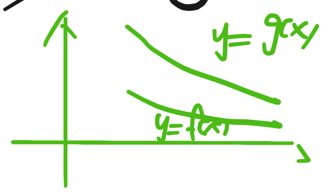
$$\tilde{h}(c) = \int_a^c f(x)\,dx \le \int_a^c g(x)\,dx = h(c)$$

$$\le M$$

$$\Rightarrow \int_a^\infty f(x)\,dx = \lim_{c\to\infty}\int_a^c f(x)\,dx \quad \text{converges} \ \#$$

<u>Thm</u>

Assume $f, g \in C^0[a,\infty)$, $f(x), g(x) > 0 \ \forall x \ge a$.



If $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $\quad 0 < L < \infty$

<u>Recall</u> "big-oh" "small-oh"

then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$

either both converge or both diverge.

pf

Step 1 $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)}=L$

$\Rightarrow$ for $\varepsilon=\frac{L}{2}>0$, $\exists\ K\geq a$ s.t.

$$\left|\frac{f(x)}{g(x)}-L\right|<\frac{L}{2}\qquad\forall x\geq K$$

$$\frac{L}{2}<\frac{f(x)}{g(x)}<\frac{3}{2}L$$

$$0<\frac{L}{2}\cdot g(x)<f(x)<\frac{3}{2}L\cdot g(x)$$

Step 2 Assume $\int_a^\infty g(x)\,dx$ Converges. Then

$$\tilde{h}(c)=\int_a^c f(x)\,dx\qquad\forall c\geq K$$

$$=\int_a^K f(x)\,dx+\underline{\int_K^c f(x)\,dx}$$

$f(x)>\frac{L}{2}g(x)>0$

$\int_K^c f(x)\,dx\leq\int_K^c\frac{3L}{2}g(x)\,dx\leq M$ for some $M\in\mathbb{R}$ $\because\int_a^\infty g(x)\,dx$ converges

is increasing when $c\geq K$

and $\tilde{h}(c)$ is bounded by

$$\int_a^k f(x)\,dx + M$$

$\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ converges

~~Step3~~

If $\int_a^{\infty} f(x)\,dx$ converges, then

since $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{L}$, $0 < \frac{1}{L} < \infty$

(Step1 + Step2)

by the previous argument, we can conclude

$\int_a^{\infty} g(x)\,dx$

also converges. #

Example

① $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \forall\, x \geq 1$$

and

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{Converges} \qquad \#$$

② $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-\frac{1}{2}}}\,dx$ diverges because

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2-\frac{1}{2}}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{2x^2}}} = 1 \qquad \infty > 1 > 0$$

and

$$\int_1^\infty \frac{1}{x}\,dx \quad \text{diverges} \qquad \#$$

③ $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ Converges because

$$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = 1 \qquad 0 < 1 < \infty$$

and

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{Converges} \qquad \#$$

④ $\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$ diverges because

$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x\to\infty} 1-e^{-x} = 1 \underset{0<}{<\infty}$$

and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges. #

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§ Differential equations

A differential equation is an equation which contains unknown functions and their derivatives. For example,

① $y'(x) = y(x)$ ← order 1

② $y''(x) = x^2 + y(x)$ ← order 2

are differential equations.

The order of a diff. eq. is the highest order of derivatives in the eq.

An ordinary differential equation (ODE) is a diff. eq. which consists of functions of one variable.

A partial differential equation (PDE) is a diff. eq. which consists of functions of many variables and their partial derivatives.

Remark

The solutions of

$$y'(x) = f(x)$$

are

$$y(x) = \int' f(x)dx = \int_a^{\cdot} f(t)dt + C$$

where C is an arbitrary constant.

Thus, a diff. eq. may have infinitely many solutions

The general solution of a diff. eq. consists of all the solutions of this diff eq.

A particular solution of a diff eq. is a certain special solution of this eq.

e.g Consider $y' = x$

$y = \frac{x^2}{2}$ is a particular sol,

$y = \frac{x^2}{2} + C$ is the general sol.

Recall

Thm $k \in \mathbb{R}$

If $f \in C^1(a,b)$ and

$$\circledast \quad f'(x) = k \cdot f(x) \quad \forall x \in (a,b),$$

a first order ODE

then $\exists C \in \mathbb{R}$ s.t.

$$f(x) = C \cdot e^{kx} \quad \forall x \in (a,b).$$

Pf

$(e^{-kx})'$

By $\circledast$,

$$e^{-kx}\left(f'(x) - k \cdot f(x)\right) = 0$$

$$\parallel$$

$$e^{-kx} \cdot f'(x) + (e^{-kx})' \cdot f(x)$$

$$\parallel$$

$$\left(e^{-kx} \cdot f(x)\right)'$$

$\Rightarrow \exists C \in \mathbb{R}$ st.

$$e^{-kx} \cdot f(x) = C \quad \forall x \in (a,b)$$

$$\Rightarrow \quad f(x) = C \cdot e^{kx} \quad \forall x \in (a,b)$$

#

This method (積分因子) applies to

$$\boxed{y'(x) + \underline{p(x)} \cdot y(x) = g(x)}$$

NOTE:

$$\left(e^{f(x)}\right)' = f'(x) \cdot e^{f(x)}$$

Sol

Consider

$$H(x) = \int p(x)\, dx$$

H(x) is any function s.t. $H'(x) = p(x)$

and

$$e^{H(x)}$$

Then by the eq,

$$e^{H(x)} \cdot (y' + p(x)\, y) = e^{H(x)} \cdot g(x)$$

$$\Vert \qquad = \left(e^{H(x)}\right)' \qquad \Vert$$

$$e^{H(x)} y' + \boxed{p(x)\, e^{H(x)}} \cdot y = \left(e^{H(x)} \cdot y\right)'$$

$$\Rightarrow \quad e^{H(x)} \cdot y = \int e^{H(x)} \cdot g(x)\, dx$$

$$\Rightarrow \quad y = e^{-H(x)} \cdot \int e^{H(x)} \cdot g(x)\, dx \qquad \#$$

(§9.1)

Example

Solve

$$(*) \qquad y' + y = 1$$

sol

$(*)$

$$\Rightarrow \quad e^{x} \cdot (y' + y) = 1 \cdot e^{x}$$

$$\shortparallel$$

$$e^{x} \cdot y' + e^{x} y = (e^{x} \cdot y)'$$

$$\Rightarrow \quad e^{x} y = e^{x} + C \qquad \text{for some } C \in \mathbb{R}$$

$$\Rightarrow \quad y = 1 + C \cdot e^{-x} \qquad \#$$

Consider $\quad (y = y(x))$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, \cdots, a_{n-1} \in \mathbb{R}$.

For example,

$$y' + ay = 0$$

$$\Rightarrow \quad e^{+ax} y' + e^{+ax} a y = 0$$

$$\|$$

$$(e^{ax} \cdot y)'$$

$\Rightarrow \exists C \in \mathbb{R}$ s.t.

$$e^{ax} \cdot y = C$$

$$\Rightarrow \quad y = C \cdot e^{-\sin} \qquad \#$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ 0 \end{pmatrix}$$

Example $(n=2)$

① $y'' = 0 = (y')' = \underline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}\begin{pmatrix} y \\ y' \end{pmatrix}$ $(y \in C^\infty(\mathbb{R}))$

Sol

Since $(y')' = 0$, $\exists\, c_1 \in \mathbb{R}$ s.t.

$$y' = c_1$$

$$\Rightarrow \quad y(x) = \underbrace{\int_0^x y'(t)\,dt}_{y(x)-y(0)} + \underbrace{y(0)}_{c_2 \in \mathbb{R} \,//\, \text{let}}$$

$$= \int_0^x c_1 \, dt + C_2$$

$$= C_1 \cdot x + C_2 \qquad \#$$

② $y'' - y = 0 \quad (*)$

Sol

Let $z(x) = y'(x)$.

$$\Rightarrow \begin{cases} y' = z \\ z' = y'' \overset{(*)}{=} y \end{cases} \quad \text{i.e.} \quad \begin{cases} y' = z \\ z' = y \end{cases}$$

That is,

$$\begin{pmatrix} y \\ z \end{pmatrix}' \overset{\text{def}}{=} \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \quad - \text{②}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

This is of the form

$$\underset{n\times 1}{\vec{y}\,'(x)} = \underset{n\times n}{A} \cdot \underset{n\times 1}{\vec{y}(x)}$$

Guess: $\vec{y}(x) \overset{?}{=} \vec{C}\, e^{Ax}$ (matrix: Ax; $e^{Ax} = ??$)

which is similar to

$$(**) \quad f'(x) = a \cdot f(x) \iff f(x) = C \cdot e^{ax}$$

Idea: by (**), we can solve diagonal matrix

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} ay \\ bz \end{pmatrix}$$

$$\left(\Leftrightarrow \quad y = c_1 e^{ax}, \quad z = c_2 e^{bx} \right)$$

So we want to transfer ② to this simple form.

<u>Actual transfer:</u>

Recall we are solving

$$\begin{cases} y' = z & — ⓐ \\ z' = y & — ⓑ \end{cases}$$

ⓐ + ⓑ :

$$(y+z)' = y' + z' = z + y = 1 \cdot (y+z)$$

$$\Rightarrow \quad y + z = C_1 e^{x} \qquad \text{for some } C_1 \in \mathbb{R}$$

ⓐ − ⓑ :

$$(y-z)' = y' - z' = z - y = (-1) \cdot (y-z)$$

$$\Rightarrow \quad y - z = C_2 \cdot e^{-x} \quad \text{for some } C_2 \in \mathbb{R}$$

So

$$\begin{cases} y + z = C_1 e^{x} \\ y - z = C_2 e^{-x} \end{cases}$$

$$\Rightarrow \quad y = \frac{C_1 e^{x} + C_2 e^{-x}}{2}$$

$$= \underline{\tilde{C}_1 e^{x} + \tilde{C}_2 e^{-x}} \quad \text{for some } \tilde{C}_1, \tilde{C}_2 \in \mathbb{R}$$

#

general sol.

In terms of matrices,

before tranfer:

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

after:

$$\begin{pmatrix} y+z \end{pmatrix}' = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} y+z \end{pmatrix}$$

$$\begin{pmatrix} \\ y-z \end{pmatrix} = \underbrace{\begin{pmatrix} & \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} \\ y-z \end{pmatrix}$$

$$\begin{pmatrix} \\ y-z \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{Q} \begin{pmatrix} y \\ z \end{pmatrix} \right) \qquad \begin{pmatrix} \\ y-z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{Q} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$Q \begin{pmatrix} y \\ z \end{pmatrix} = A \left(Q \begin{pmatrix} y \\ z \end{pmatrix} \right)$$

Also, note that Q is invertible:

$$Q^{-1} = \frac{1}{-1-1} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

This matrix satisfies the conditions:

$$Q^{-1} \cdot Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} - \left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$Q\cdot Q^{-1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

So

$$\begin{pmatrix}y\\z\end{pmatrix}' = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}\right)'$$

$$= \left(Q^{-1}\cdot \boxed{Q\begin{pmatrix}y\\z\end{pmatrix}}\right)' = AQ\begin{pmatrix}y\\z\end{pmatrix}$$

$$= Q^{-1}AQ\begin{pmatrix}y\\z\end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Remark

$$(B\vec{y})' = B\cdot(\vec{y}')$$

$$\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}\right)'$$

$$= \begin{pmatrix}ay+bz\\cy+dz\end{pmatrix}'$$

$$= \begin{pmatrix}(ay+bz)'\\(cy+dz)'\end{pmatrix}$$

$$= (ay'+bz'$$

$$= \left(cy' + dz' \right)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ z' \end{pmatrix}$$

Summary

To solve

$$(\star) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

we do the following:

Step 1:

We introduce new unknown functions:

$$\begin{aligned} y_0 &= y \\ y_1 &= y' \\ y_2 &= y'' \\ &\vdots \\ y_{n-1} &= y^{(n-1)} \end{aligned} \qquad \text{and} \quad \vec{y} = \underset{n \times 1}{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}}$$

Then

$$\vec{y}'(x) = \begin{pmatrix} y_0' \\ y_1' \\ \vdots \\ \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{pmatrix}$$

$$\overset{\circledast}{=} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ y_{n-1} \\ -a_{n-1} y_{n-1} - a_{n-2} y_{n-2} - \cdots - a_1 y_1 - a_0 y_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & & \\ 0 & 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ \\ y_{n-1} \end{pmatrix}$$

So we get an eq. of the form

$$\underset{n\times 1}{y'} = \underset{n\times n}{A} \cdot \underset{n\times 1}{y}$$

<u>Step2</u> Diagonalization (對角化) ← Linear Algebra

Find an invertible matrix Q and a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_2 \end{pmatrix}$$

s.t.

$$Q^{-1} \cdot D \cdot Q = A$$

or

$$D = Q \cdot A \cdot Q^{-1}$$

<u>Step3</u> Assume

$$A = Q^{-1} \cdot D \cdot Q$$

Then

$$\vec{y}\,' = A\,\vec{y} = (Q^{-1} \cdot D \cdot Q) \cdot \vec{y}$$

$$\Rightarrow \left(Q\vec{y}'\right) = \cancel{Q\cdot Q^{-1}} D\cdot Q\,\vec{y} = D\left(Q\vec{y}\right)$$

$$\left(Q\vec{y}'\right) = \left(Q\vec{y}\right)'$$

Let

$$\underline{\vec{z} = Q\vec{y}} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Then

$$\vec{z}' = D\cdot\vec{z}$$

$$\vec{z}' = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

i.e.

$$\begin{cases} z_1' = \lambda_1 z_1 \\ z_2' = \lambda_2 z_2 \\ \vdots \end{cases} \Rightarrow \begin{cases} z_1 = C_1 e^{\lambda_1 x} \\ z_2 = C_2 e^{\lambda_2 x} \end{cases}$$

$$\begin{cases} \quad \vdots \\ z_n' = \lambda_n z_n \end{cases} \qquad \begin{cases} \quad \vdots \\ z_n = C_n \cdot e^{\lambda_n x} \end{cases}$$

So

$$\vec{y} = Q^{-1} \cdot \vec{z} = Q^{-1} \cdot \begin{pmatrix} C_1 e^{\lambda_1 x} \\ C_2 e^{\lambda_2 x} \\ \vdots \\ C_n e^{\lambda_n x} \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} y_0 = y \\ y_1 = y' \\ \vdots \\ y_{n-1} = y^{(n-1)} \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} C_1 e^{\lambda_1 x} + a_{12} C_2 e^{\lambda_2 x} + \cdots + a_{1n} C_n e^{\lambda_n x} \\ \\ \end{pmatrix}$$

<u>Conclusion</u>

$$y(x) = \tilde{c}_1 e^{\lambda_1 x} + \tilde{c}_2 e^{\lambda_2 x} + \cdots + \tilde{c}_n e^{\lambda_n x}$$

#

③ $y'' + y = 0$

Sol

Let $z = y'$. Then

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} z \\ -y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} y \\ z \end{pmatrix}' - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0$$

$$\left(\times \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \right)$$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}} \cdot \begin{pmatrix} y' \\ z' \end{pmatrix}$$

$$- \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0$$

$$\|$$

$$-\begin{pmatrix}\sin x & \cos x\\ -\cos x & \sin x\end{pmatrix} = \begin{pmatrix}-\sin x & -\cos x\\ \cos x & -\sin x\end{pmatrix}$$

$$= \begin{pmatrix}(\cos x)' & (-\sin x)'\\ (\sin x)' & (\cos x)'\end{pmatrix}$$

Let

$\vec{y} = \begin{pmatrix} y\\ z\end{pmatrix}$

$B(x) = \begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos y\end{pmatrix}$

$$= B(x)\cdot \vec{y}\,'(x) + B'(x)\cdot \vec{y}(x) = 0$$

$\|$ ← you can check

$$\left(B(x)\cdot \vec{y}(x)\right)'$$

That is,

$$\begin{pmatrix}(\cos x\cdot y - \sin x\, z)'\\ (\sin x\cdot y + \cos x\cdot z)'\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$\Rightarrow \exists\, c_1, c_2 \in \mathbb{R}$

$$\cos x\cdot y - \sin x\, z = c_1$$

$$\sin x\cdot y + \cos x\cdot z = c_2$$

general sol.

(all the sol

$$\Rightarrow \quad y = C_1 \cos x + C_2 \cdot \sin x$$ ↙ one of this form

$$\left(z = C_1(-\sin x) + C_2 \cos x \right)$$

#

Summary:

Thm (Thm 9.3.6)

Consider

$$y'' + ay' + by = 0 \quad , \quad a, b \in \mathbb{R}.$$

To solve, consider its "characteristic eq":

"det(A - rI) = 0"

$$\text{(C)} \quad r^2 + ar + b = 0$$

(i) If (C) has two different real roots $r = r_1, r_2$, then the general sol is

$$y = C_1 \cdot e^{r_1 x} + C_2 \cdot e^{r_2 x}$$

eg $y'' - y = 0 \rightsquigarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$\Rightarrow y = C_1 e^{x} + C_2 e^{-x}$$

$r = \alpha$

(ii) If ⓒ has exactly one real root r,
then

$$y = (C_1 + C_2 x) \cdot e^{\alpha x}$$

e.g. $y'' = 0 \rightsquigarrow r^2 = 0 \Rightarrow r = 0$

$$\Rightarrow y = (C_1 + C_2 x) \cdot e^{0 \cdot x} = C_1 + C_2 x$$

(iii) If ⓒ has 2 nonreal roots

$$r = \alpha \pm \beta \sqrt{-1} \qquad (\alpha, \beta \in \mathbb{R}),$$

then

$$y = (C_1 \cos \beta x + C_2 \sin \beta x) \, e^{\alpha x}$$

e.g. $y'' + y = 0 \longrightarrow r^2 + 1 = 0 \Rightarrow r = \pm \sqrt{-1}$ $\quad \alpha = 0, \beta = 1$

$$\Rightarrow \quad y = (C_1 \cos x + C_2 \sin x) \cdot e^{0 \cdot x}$$

$$= C_1 \cos x + C_2 \sin x$$

Here $C_1, C_2 \in \mathbb{R}$

Thm

To solve

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

consider the char. eq.

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Use (i)–(iii) in the previous thm, you can find n "linearly independent" sol's

$$x^k \cdot e^{r_j x}, \quad x^k e^{\alpha_j x} \cos \beta_j x, \quad x^k e^{\alpha_j x} \sin \beta_j x$$

Then

$$y = C_1 \boxed{} + C_2 \boxed{} + \cdots + C_n \boxed{}$$

Calculus — Quiz 5 (Fall 2025)

Name: ________________ Student ID: ________________ Score: ______ (Total of points is 60.)

1. (20 points) Prove that for $x > 0$,

$$x - \frac{x^2}{2} < \ln(1 + x) < x.$$

1. $x-\frac{x^2}{2} < \log(1+x) < x$ | $f' < g' \Rightarrow f' - g' < 0$

$\Rightarrow (f(x) - g(x)) - (f(0) - g(0))$

$\exists c$ s.t. $= f'(c) - g'(c) > 0$

2. Calculates.

   (a) (10 points) $\displaystyle\int_0^1 \frac{e^x}{4 - e^x}\,dx.$

   (b) (10 points) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{1 + \sin x}\,dx.$

這次quiz 5的主要問題如下

第一題：f=x-x^2/2, g=log(1+x), h=x

A. 三者的導函數具有「f'<g'<h'」的關係，所以直接推論「f<g<h」。（應該要再加上f,g,h在0點的等值關係）

B. 同樣由三者導函數的關係進行「不定」積分，所以推出結論。（這樣子會有不同的積分常數問題）

C. 同樣由三者導函數的關係進行「定」積分，但是積分範圍從1開始到x，然後也忽略掉下邊緣的函數值不相等，所以推出結論。

D. 利用積分法論證，我會要求他們除了導函數的關係外，還要再註明導函數們的「連續性」，這樣的正則性保證積分的嚴格不等式。

$f(x) = \int_1^x f'(t)\,dt \boxed{+ f(1)}$

第二題：

基本上沒有太大問題，頂多就是計算時的小錯誤

A. 變數變換忘記上下限跟著動。

B. 1/x不會積分。

C. 這個是比較誇張的，f/(g+h)的積分是F/(G+H)，其中大寫字母為對應小寫字母的primitive function。

$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$

$\left(\frac{F}{G+H}\right)' = \frac{F'\cdot(G+H) - F\cdot(G+H)'}{(G+H)^2} = \frac{f(G+H) - F(g+h)}{(G+H)^2} \neq \frac{f}{g+h}$

第三題：

A. 使用L-Hospital的時候要註明使用時的形式，例如：(0/0)。

B. 有些人的e^x的微分是xe^{x-1}。

以上是quiz 5 的常見錯誤部分

$\boxed{(e^x)' = e^x}$

3. (20 points) Prove that, if n is a positive integer, then there exists N such that

$$e^x > x^n$$

for all $x \geq N$.