# Calculus, Fall 2025, week 14

(§8.4)

Example

① $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = ?$ $(a>0)$

Recall

$\cos^2\theta + \sin^2\theta = 1$

$\Downarrow$

$\cos^2\theta = 1-\sin^2\theta$

sol $\left(u = \sin^{-1}\left(\frac{x}{a}\right)\right)$

Let $x = a\sin u \Rightarrow dx = a\cos u\,du$

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \int_{\underbrace{u(-a)}_{\sin^{-1}(-1) = -\frac{\pi}{2}}}^{\overbrace{u(a)}^{\frac{\pi}{2}}} \underbrace{\sqrt{a^2-a^2\sin^2 u}}_{\begin{aligned} &= a\sqrt{1-\sin^2 u} = a|\cos u| \\ &= a\cos u\end{aligned}}\, a\cos u\,du$$

$\cos u \geq 0$
$\forall u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{a\cos u \cdot a\cos u\,du}$$

Recall

$\cos 2u$

$= \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^2 u}_{=\frac{1+\cos 2u}{2}}\,du$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\frac{1}{2}}_{=\left(\frac{u}{2}\right)'} + \underbrace{\frac{1}{2}\cos 2u}_{=\left(\frac{\sin 2u}{4}\right)'}\,du$$

$$= a^2\left(\frac{u}{2} + \frac{\sin 2u}{4}\right)\Bigg|_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{2}a^2 \quad \#$$

② $\int \sqrt{a^2+x^2}\,dx = ?$ $(a>0)$

Recall

$\rightarrow 1+\tan^2\theta = \sec^2\theta$

sol $\left(u = \tan^{-1}\frac{x}{a}\right)$

Step 1

Let $x = a\tan u \Rightarrow dx = a\sec^2 u\,du$

$a\sec u$

$=$

$$\Rightarrow \int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\tan^2 u}\cdot a\sec^2 u\,du$$

$$= a^2 \int \sec^3 u\,du$$

Recall: $\int uv'dx = uv - \int vu'dx$

Step 2 $\int \sec^3 u\,du = ?$

$(\tan u)'$

$\parallel$

$\left(\left(\frac{1}{\cos u}\right)' = \frac{-(-\sin u)}{\cos^2 u}\right)$

$$\int \sec^3 u\,du = \int \sec u\cdot \sec^2 u\,du$$

$\tan u \sec u$

$$= \sec u\cdot\tan u - \int (\sec u)'\cdot\tan u\,du$$

$\sec^2 u - 1$

$\parallel$

$$= \sec u\cdot\tan u - \int \sec u\ \tan^2 u\ du$$

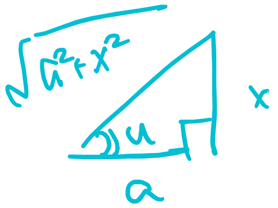
$$= \sec u \tan u - \int \sec u \tan^2 u \, du$$

$$= \sec u \cdot \tan u - \int \sec^3 u \, du + \int \sec u \, du$$

$$\Rightarrow \int \sec^3 u \, du = \frac{1}{2}\left(\sec u \tan u + \int \sec u \, du\right) + C$$

$$= \frac{1}{2}\left(\sec u \tan u + \log\left|\sec u + \tan u\right|\right) + C$$

**Step3** $\tan u = \frac{x}{a}$




$\Rightarrow \sec u = \frac{\sqrt{a^2+x^2}}{a}$

$$\int \sqrt{a^2+x^2} \, dx = a^2 \int \sec^3 u \, du$$

$$= \frac{a^2}{2}\left(\sec u \tan u + \log\left|\sec u + \tan u\right|\right) + C'$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a} \cdot \frac{x}{a} + \log\left|\frac{\sqrt{a^2+x^2} + x}{a}\right|\right) + C'$$

$$\text{"} \log\left|\sqrt{a^2+x^2} + x\right| - \log a$$

$$= \frac{1}{2} x \sqrt{a^2+x^2} + \frac{a^2}{2} \log\left|\sqrt{a^2+x^2} + x\right| + C''$$

#

③ $$\int \frac{1}{\sqrt{x^2-1}}\,dx = ?$$

Recall

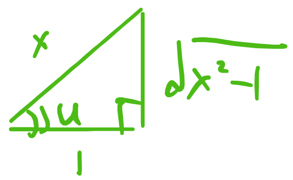
$\tan^2\theta = \sec^2\theta - 1$

Sol $(u = \sec^{-1} x)$

Let $x = \sec u \Rightarrow dx = \tan u \sec u \, du$

$$\Rightarrow \int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{1}{\underbrace{\sqrt{\sec^2 u - 1}}_{\tan u}} \cancel{\tan} u \sec u \, du$$

$$= \int \sec u \, du$$




$$= \log|\underbrace{\sec u}_{x} + \underbrace{\tan u}_{\sqrt{x^2-1}}| + C$$

$$= \log|x + \sqrt{x^2-1}| + C$$ #

Consider (§ 8.5)

$$\int \frac{P(x)}{Q(x)}\,dx$$

$\int \quad Q(x)$

where $P(x), Q(x)$ are polynomials

Step1 If $\deg P(x) \geq \deg Q(x)$, then

by the division algorthim,

$$\frac{P(x)}{Q(x)} = \frac{Q(x) g(x) + r(x)}{Q(x)} = g(x) + \frac{r(x)}{Q(x)}$$

for some polynomials $g(x), r(x)$, $\deg r(x) < \deg Q(x)$

Example

$P(x) = x^3 + x^2 + x + 1$, $\quad Q(x) = x - 1$

$$\int \frac{x^3 + x^2 + x + 1}{x - 1} \, dx = ?$$

Sol

$$\begin{array}{r|l} & x^2 + 2x + 3 \\ \hline x - 1 & x^3 + x^2 + x + 1 \\ & x^3 - x^2 \\ \hline & 2x^2 + x \\ & 2x^2 - 2x \\ \hline & 3x + 1 \end{array}$$

$$\frac{3x-3}{4}$$

$$\Rightarrow \quad x^3+x^2+x+1 = (x-1)(x^2+2x+3) + 4$$

$$\left(\frac{x^3}{3}+x^2+3x\right)' \qquad \qquad (4\log|x-1|)'$$

$$\Rightarrow \quad \int \frac{x^3+x^2+x+1}{x-1}\,dx = \int \underbrace{(x^2+2x+3)}_{} + \frac{4}{x-1}\,dx$$

$$= \frac{x^3}{3}+x^2+3x+4\log|x-1| + C$$

\#

Step 2: By Step 1, we may assume $\deg P(x) < \deg Q(x)$

Idea:

Factorize $Q(x)$ and decompose $\dfrac{P(x)}{Q(x)}$

因式分解

Example

$P(x) = 1$, $Q(x) = x^2-2x-3 = (x-3)(x+1)$

$$\int \frac{1}{\quad}\,dx = ?$$

$$\int \frac{\quad}{x^2-2x+3}$$

**Sol** Solve $A, B$ in the following

$$\frac{1}{x^2-2x+3} = \frac{0\cdot x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\frac{(A+B)x + (A-3B)}{(x-3)(x+1)} = \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

$$\Rightarrow \begin{cases} A+B = 0 \\ A-3B = 1 \end{cases} \Rightarrow A = \frac{1}{4},\ B = \frac{-1}{4}$$

$$\Rightarrow \frac{1}{x^2-2x+3} = \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}$$

$$\Rightarrow \int \frac{1}{x^2-2x+3}\,dx = \frac{1}{4}\int\frac{1}{x-3}\,dx - \frac{1}{4}\int\frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\log|x-3| - \frac{1}{4}\log|x+1| + C$$

#

$$= \frac{1}{4}\log\left|\frac{x-3}{x+1}\right| + C$$

Remark polynomial of real coefficient

$$Q(x) = a(x-b_1)^{r_1}\cdot(x-b_2)^{r_2}\cdots(x-b_i)^{r_i}\cdot(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}\cdots(x^2+c_jx+d_j)^{r_j}$$

where $a, b_i, c_j, d_k \in \mathbb{R}$, $x^2+c_kx+d$ are irreducible real polynomials
不可分解

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left(\frac{f_1(x)}{(x-b_1)^{r_1}} + \cdots + \frac{f_i(x)}{(x-b_i)^{r_i}} + \frac{f_{i+1}(x)}{(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}} + \cdots + \frac{f_j(x)}{(x^2+c_jx+d_j)^{r_j}}\right)$$

where $\deg f_k(x) < r_k$ if $k = 1,\dots i$

$\deg f_k(x) < 2r_k$ if $k = i+1, \dots, j$

<u>Example</u>

$$\frac{x+2}{x^3-1} = \frac{x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{Ax^2+Ax+A+Bx^2+Cx-Bx-C}{(x-1)(x^2+x+1)}$$

$x^2$-term: $0 = A+B$ $\qquad A=1 \quad B=-1$

$x$-term: $\quad 1 = A + C - B \quad \Rightarrow \quad C = -1$

$x^0$-term: $\quad 2 = A - C$

$$\Rightarrow \quad \frac{x+2}{x^3-1} = \frac{1}{x-1} + \frac{-x-1}{x^2+x+1} \quad \#$$

$$\Rightarrow \quad \int \frac{x+2}{x^3-1}\,dx = \underbrace{\int \frac{1}{x-1}\,dx}_{\log|x-1|} - \underbrace{\int \frac{x+1}{x^2+x+1}\,dx}_{?}$$

? ← next question

By this decomposition, we shall consider

$\boxed{\text{case 1}}$ $$\int \frac{f(x)}{(x-b)^r}\,dx \quad (\deg f(x) < r)$$

and

$\boxed{\text{case 2}}$ $$\int \frac{f(x)}{(x^2+cx+d)^r}\,dx \quad (\deg f(x) < 2r)$$

Case 1 by 除法

Decompose:

$$f(x) = g(x)\cdot(x-b) + A_r$$

$$\Rightarrow \frac{f(x)}{(x-b)^r} = \frac{g(x)\cdot\cancel{(x-b)}}{(x-b)^{\cancel{r}\ r-1}} + \frac{A_r}{(x-b)^r} = \cdots$$

$$\frac{f(x)}{(x-b)^r} = \frac{A_1}{x-b} + \frac{A_2}{(x-b)^2} + \cdots + \frac{A_r}{(x-b)^r}$$

where $A_k \in \mathbb{R}$

Recall

$$\int \frac{A}{(x-b)^r}\,dx = \begin{cases} A\log|x-b| + C & r=1 \\ \dfrac{A(x-b)^{-r+1}}{-r+1} + C & r\neq 1 \end{cases}$$

Example

$$\int \frac{x+1}{(x-1)^2}\,dx = ?$$

Sol

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{Ax+(B-A)}{(x-1)^2}$$

$$\Rightarrow \quad A=1, \quad B-A=1 \quad \Rightarrow \quad A=1,\ B=2$$

$$(\log|x-1|)' \qquad \left(\frac{-2}{x-1}\right)'$$

$$\Rightarrow \int \frac{x+1}{(x-1)^2}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2}\right)dx$$

$$= \log|x-1| - \frac{2}{x-1} + C$$

\#

<u>case 2</u>

$(c^2 - 4d < 0$
$\deg f(x) < 2r)$

$$\int \frac{f(x)}{(x^2+cx+d)^r}\,dx = ?$$

Decompose:

$$\frac{f(x)}{(x^2+cx+d)^r} = \frac{A_1x+B_1}{(x^2+cx+d)^1} + \frac{A_2x+B_2}{(x^2+cx+d)^2} + \cdots$$

$$+ \frac{A_rx+B_r}{(x^2+cx+d)^r}$$

where $A_k, B_k \in \mathbb{R}$

<u>Example</u>

$$\frac{x^3}{(x^2+1)^2} = \frac{(A_1x+B_1)(x^2+1)}{(x^2+1)^2} + \frac{A_2x+B_2}{(x^2+1)^2}$$

$$= \frac{\overset{1}{\overset{\shortparallel}{A_1}}x^3 + \overset{0}{\overset{\shortparallel}{B_1}}x^2 + \overset{0}{\overset{\shortparallel}{(A_1+A_2)}}x + \overset{0}{\overset{\shortparallel}{(B_1+B_2)}}}{(x^2+1)^2}$$

$$\Rightarrow A_1 = 1,\ B_1 = 0,\quad A_2 = -1,\ B_2 = 0$$

So

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$\frac{x}{x^2+1} = \left(\frac{1}{2}\log|x^2+1|\right)' \qquad \frac{x}{(x^2+1)^2} = \left(\frac{(x^2+1)^{-1}}{-2}\right)'$$

$$= \frac{1}{2}\log|x^2+1| + \frac{1}{2}(x^2+1)^{-1} + C \quad \#$$

Consider

$$\int \frac{Ax+B}{(x^2+cx+d)^r}\,dx = ?$$

Recall: $\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$

Case 2-1: $r=1,\ A=0$

$$\int \frac{B}{x^2+cx+d}\,dx = \int \frac{B}{(x+\frac{c}{2})^2 + (d-\frac{c^2}{4})}\,dx$$

$$d-\frac{c^2}{4} = \frac{4d-c^2}{4} > 0$$

$$= \frac{4B}{4d-c^2}\int \frac{1}{\left(\frac{2x+c}{\ldots}\right)^2 + 1}\,dx$$

Let $u = \frac{2x+c}{\sqrt{4d-c^2}} \Rightarrow du = \frac{2}{\sqrt{4d-c^2}} dx$

$$= \frac{2 \cancel{4} B}{\sqrt{4d-c^2}} \int \frac{1}{u^2+1} \cdot \frac{\cancel{\sqrt{4d-c^2}}}{\cancel{2}} du$$

$$= \frac{2B}{\sqrt{4d-c^2}} \tan^{-1} u + C'$$

$$= \frac{2B}{\sqrt{4d-c^2}} \tan^{-1}\left(\frac{2x+c}{\sqrt{4d-c^2}}\right) + C'$$

#

Example

$$\int \frac{1}{x^2+x+1} dx = \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx$$

$$= \frac{4}{3} \int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx$$

$$u = \frac{\phantom{x}}{\sqrt{3}} \Rightarrow du = \frac{\phantom{x}}{\sqrt{3}}\, dx$$

$$= \cancel{\frac{4}{3}}^{\frac{2}{\sqrt{3}}} \int \frac{1}{u^2+1} \cancel{\frac{\sqrt{3}}{2}}\, du$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} u + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C \quad \#$$

Case 2-2: $r=1,\ A \neq 0.$

$$\int \frac{Ax+B}{\underbrace{x^2+Cx+d}_{\downarrow \frac{d}{dx} \ 2x+C}}\, dx = \int \frac{\frac{A}{2}(2x+C)}{x^2+Cx+d} + \frac{B-\frac{AC}{2}}{x^2+Cx+d}\, dx$$

$$= \frac{A}{2} \int \underbrace{\frac{2x+C}{x^2+Cx+d}}_{\parallel \ \left(\log|x^2+Cx+d|\right)'} dx + \left(B - \frac{AC}{2}\right) \underbrace{\int \frac{1}{x^2+Cx+d}\, dx}_{\uparrow \text{ by case 2-1}}$$

$$\left(\log|x^2+x+1|\right)'$$

$$\parallel$$

## Example

$$\int \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{-\frac{1}{2}}{x^3+x+1}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$\#$

**case 2-3** $r>0$ and $A=0$.

$$\int \frac{B}{(x^2+cx+d)^r}\,dx = \int \frac{B}{\left((x+\frac{c}{2})^2+(d-\frac{c^2}{4})\right)^r}\,dx$$

$$= \frac{B}{(d-\frac{c^2}{4})^r}\int \frac{1}{\left(\left(\frac{2x+c}{\sqrt{4d-c^2}}\right)^2+1\right)^r}\,dx$$

$2x+c$

Let $u = \frac{}{\sqrt{4d-c^2}} \Rightarrow du = \frac{}{\sqrt{4d-c^2}}\,dx$

$$= \frac{B}{\left(d-\frac{c^2}{4}\right)^r} \int \frac{1}{(u^2+1)^r} \frac{\sqrt{4d-c^2}}{2}\,du$$

Let $u = \tan v \Rightarrow du = \sec^2 v\,dv$

$(v = \tan^{-1} u)$

Recall:

$1+\tan^2\theta = \sec^2\theta$

$$= \frac{1}{\sec^{2r} v}$$

$$= \frac{B\sqrt{4d-c^2}}{2\left(d-\frac{c^2}{4}\right)^r} \int \frac{1}{(\tan^2 v+1)^r} \sec^2 v\,dv$$

$$= \frac{B\sqrt{4d-c^2}}{2\left(d-\frac{c^2}{4}\right)^r} \int \cos^{2r-2} v\,dv = \cdots$$

Example

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \int \frac{1}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2}\,dx$$

$u = \frac{2x+1}{\sqrt{3}}$

$$= \int \frac{1}{}$$

$$= \frac{16}{9}\int \frac{1}{\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)^2}\,dx \qquad du = \frac{2}{\sqrt{3}}\,dx$$

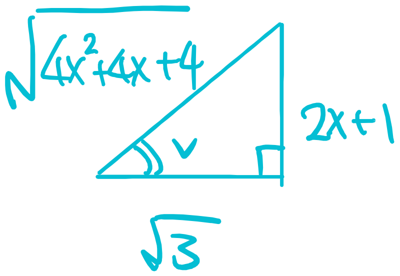
$$= \frac{8\sqrt{3}}{9}\int \frac{1}{(u^2+1)^2}\,du \qquad \begin{array}{l} v = \tan^{-1} u \\ \text{or } u = \tan v \\ du = \sec^2 v\, dv \end{array}$$

$$\cos^2 v = \frac{\cos 2v + 1}{2}$$

$$= \frac{8\sqrt{3}}{9}\int \frac{1}{(\underbrace{\tan^2 v+1}_{\sec^2 v})^2}\,\sec^2 v\, dv$$

$$u = \frac{2x+1}{\sqrt{3}} = \tan v$$

$$= \frac{\overset{4}{\cancel{8}}\sqrt{3}}{9}\int \frac{\cos 2v+1}{\cancel{2}}\,dv$$




$$= \frac{4\sqrt{3}}{9}\left(\frac{\sin 2v}{2} + v\right) + C$$

$$= \frac{4\sqrt{3}}{9}\left(\underbrace{\sin v}_{\frac{2x+1}{2\sqrt{x^2+x+1}}}\cdot\underbrace{\cos v}_{\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}} + \underbrace{v}_{=\tan^{-1}\frac{2x+1}{\sqrt{3}}}\right) + C$$

$$= \frac{4\sqrt{3}}{9}\Big( \sqrt{3}(2x+1) \quad \tan^{-1}(2x+1 \quad \Big) + C$$

$$= \frac{}{9}\left( \frac{}{4(x^2+x+1)} + \tan^{-1}\left(\frac{}{\sqrt{3}}\right)\right) + C$$

#

<u>case 2-4</u> $r>1$, $A\neq 0$:

by <u>case 2-3</u>

$$\int \frac{Ax+B}{(x^2+cx+d)^r}\,dx = \int \frac{\frac{A}{2}(2x+c)}{(x^2+cx+d)^r} + \frac{B-\frac{Ac}{2}}{(x^2+cx+d)^r}\,dx$$

$$\frac{\frac{A}{2}(2x+c)}{(x^2+cx+d)^r} = \left(\frac{\frac{A}{2}}{-r+1}(x^2+cx+d)^{-r+1}\right)'$$

$$-\left(\frac{1}{x^2+x+1}\right)'$$

<u>Example</u>

$$\int \frac{x}{(x^2+x+1)^2}\,dx = \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)^2}\,dx + \int \frac{-\frac{1}{2}}{(x^2+x+1)^2}\,dx$$

|| by previous example

$$= -\frac{1}{2}\cdot\frac{1}{x^2+x+1} - \frac{1}{2}\left(\frac{1}{3}\,\frac{2x+1}{x^2+x+1} + \frac{4\sqrt{3}}{9}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right)$$

$$+ C$$

#

瑕積分

# Improper integrals

**Def** (§11.7)

Integrals with infinite upper/lower bounds are called improper integrals (of type I)

(i) If $f \in C^0[a,\infty)$, then

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

(ii) If $f \in C^0(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx := \lim_{a\to-\infty} \int_a^b f(x)dx$$

(iii) If $f \in C^0(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)dx := \int_c^\infty f(x)dx + \int_{-\infty}^c f(x)dx$$

where $c$ is any real number.

(and finite)

In each case, if the limit exists (and finite), we say the improper integral converges, and the limit is the value of the improper integral.
If the limit does not exist, we say the improper integral diverges.

Example

① $$\int_1^\infty \frac{\log x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\log x}{x^2}\,dx$$

$$\int_1^b \underbrace{\frac{1}{x^2}}_{v'}\cdot\underbrace{\log x}_{u}\,dx = \frac{-1}{x}\cdot\log x\Big|_1^b - \int_1^b \frac{-1}{x}\cdot\frac{1}{x}\,dx$$

$v = \frac{-1}{x}$

$$\frac{-1}{x}\Big|_1^b = -\frac{1}{b}+1$$

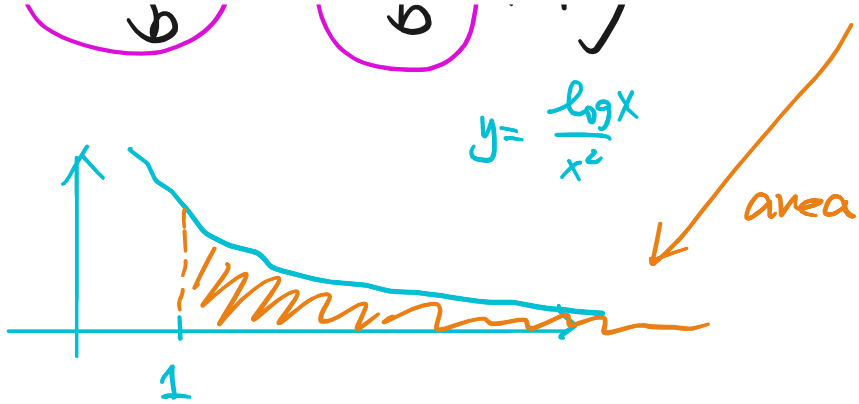
$$= \lim_{b\to\infty}\left(-\frac{1}{b}\log b + 0 + \int_1^b \frac{1}{x^2}\,dx\right)$$

$$\lim_{b\to\infty}\frac{-\frac{1}{b}}{1} = 0$$

$0$

$$= \lim\left(-\frac{\log b}{b} - \frac{1}{b} + 1\right) = 1$$

$b \to \infty$

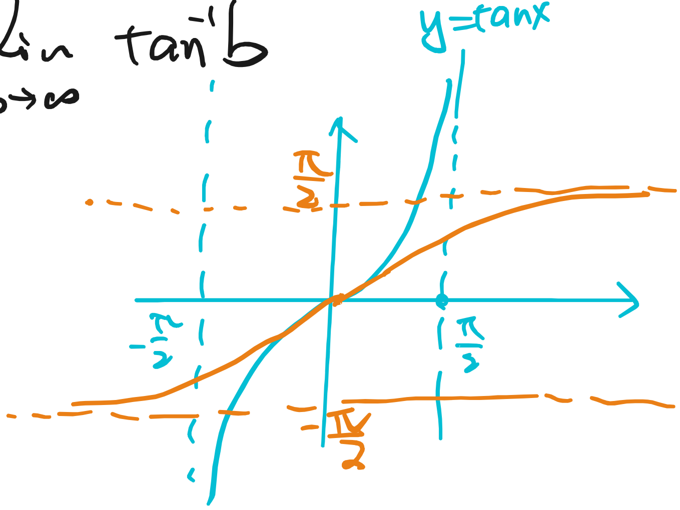



② $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \Big( \underbrace{\tan^{-1}x \Big|_{0}^{b}}_{\tan^{-1}b \,-\, \underbrace{\tan^{-1}0}_{0}} \Big)$$

$$= \lim_{b\to\infty} \tan^{-1} b$$

$$= \frac{\pi}{2}$$




$y = \tan^{-1} x$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_{x=a}^{0} = \lim_{a\to-\infty} \left(-\tan^{-1} a\right)$$

$u \to -\infty$ $\qquad\qquad$ $u \to -\infty$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \#$$

③ $$\int_1^{\infty} \frac{1}{x^p}\,dx = ?$$

$$= \lim_{b\to\infty} \boxed{\int_1^b \frac{1}{x^p}\,dx} = \begin{cases} \left.\dfrac{x^{-p+1}}{-p+1}\right|_{x=1}^{b} = \dfrac{b^{-p+1}}{-p+1} - \dfrac{1}{-p+1} & p \neq 1 \\ \log x \Big|_1^b = \log b & p = 1 \end{cases}$$

<u>case 1:</u> $p > 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} + \frac{1}{p-1}\right) = \frac{1}{p-1}$$

$-p+1 < 0$, $\to 0$

<u>case 2:</u> $p < 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} + \frac{1}{p-1}\right)$$

$-p+1 > 0$

diverges does NOT exist

Case 3: $p=1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \log b \quad \text{does NOT exist}$$

diverges

Conclusion

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p>1 \\ \text{diverges}, & p\le 1 \end{cases}$$

#

Def

Integrals of functions that become infinite at a point in the interal of integration are called improper integrals (of type II)

(i) If $f \in C^0(a,b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$,

then

$$\int_a^b f(x)\,dx := \lim_{c \to a^+} \int_c^b f(x)\,dx$$

(ii) If $f \in C^0[a,b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$,

then

$$\int_a^b f(x)\,dx := \lim_{c \to b^-} \int_a^c f(x)\,dx$$

(iii) If $f \in C^0([a,c) \cup (c,b])$ and $\lim_{x \to c} f(x) = \pm\infty$

then

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit exists, we say the improper integral converges

Otherwist, we say it diverges