Calculus, Fall 2025, week 13

Recall

- $\log : (0, +\infty) \xrightarrow{1-1,\text{ onto}} \mathbb{R}$

  $\log x := \int_1^x \frac{1}{t}\,dt \quad \Rightarrow (\log x)' = \frac{1}{x}$

- $\exp := (\log)^{-1} : \mathbb{R} \to (0, +\infty)$

  $\exp(\log y) = y \quad \forall\, y \in (0, \infty)$

  $\log(\exp x) = x \quad \forall\, x \in \mathbb{R}$

  $\Rightarrow \boxed{(\exp x)' = \exp x}$ $\Rightarrow (e^{ax})' = a \cdot e^{ax}$

- $\exp(x) = e^x$, where $e = \log^{-1}(1)$ (or $\log e = 1$)

  $\log y = \log_e y$

Example

① $\int_0^1 9e^{3x}\,dx = 3e^{3x}\Big|_0^1 = 3e^3 - 3$ #

② $\int_0^1 \frac{e^{3x}\,dx}{e^{3x}+1}$ $\qquad \frac{1}{3}du$

$u = e^{3x}+1 \Rightarrow du = 3e^{3x}dx$

$= \int_{u(0)=2}^{u(1)=e^3+1} \frac{1}{u}\,\frac{1}{3}\,du = \frac{1}{3}\log u \Big|_2^{e^3+1}$

$= \frac{1}{3}\left(\log(e^3+1) - \log 2\right) = \frac{1}{3}\log\left(\frac{e^3+1}{2}\right)$ #

③ $\int_0^1 x e^x\,dx$

$(e^x)'$

$\int_a^b u(x)v'(x)\,dx = u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$

$u(x) = x \qquad u'(x) = 1$

$v(x) = e^x \qquad v'(x) = e^x$

$= xe^x\Big|_0^1 - \int_0^1 e^x\,dx$

$= xe^x\Big|_0^1 - e^x\Big|_0^1$

$= e - (e-1) = 1$ #

Thm (Thm 7.6.1)

If

$(k \in \mathbb{R})$

(*) $f'(x) = k\cdot f(x) \quad \forall x \in (a,b)$

then $\exists\, C \in \mathbb{R}$ s.t.

$$f(x) = C \cdot e^{kx} \qquad \forall x \in (a,b)$$

pf

$(*) \Rightarrow \quad f'(x) - k f(x) = 0$

$$\Rightarrow \quad e^{-kx} \cdot \left(f'(x) - k f(x) \right) = 0$$

$$\| \qquad\qquad = \left(e^{-kx}\right)'$$

$$e^{-kx} f'(x) - k e^{-kx} f(x)$$

$$\| \qquad\qquad \forall x \in (a,b)$$

$$\left(e^{-kx} f(x) \right)'$$

$\Rightarrow \quad \exists\, C \in \mathbb{R}$ s.t.

$$e^{-kx} f(x) = C \qquad \forall x \in (a,b)$$

$$\Rightarrow \quad f(x) = C \cdot e^{kx} \qquad \#$$

Thm (§11.4) $\quad a_n$

$$\lim_{n\to\infty} \boxed{\left(1+\frac{x}{n}\right)^n} = e^x$$

In particular $(x=1)$,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

pf

For $x=0$, $\quad LHS = 1 = RHS$

For $x\neq 0$,

$$\log\left(\left(1+\frac{x}{n}\right)^n\right) = n\cdot\log\left(1+\frac{x}{n}\right)$$

$$= x\cdot \boxed{\frac{\log\left(1+\frac{x}{n}\right) - \log 1}{\frac{x}{n}}} \to (\log t)'\Big|_{t=1}$$

$$\updownarrow$$

$$(\log t)'\Big|_{t=1} = \lim_{h\to 0}\frac{\log(1+h) - \log 1}{h}, \quad h = \frac{x}{n}$$

$\Rightarrow$

$$\lim_{n\to\infty} \log\left(\left(1+\frac{x}{n}\right)^n\right) = x \cdot \lim_{n\to\infty} \frac{\log\left(1+\frac{x}{n}\right) - \log 1}{\frac{x}{n}}$$

$$= x \cdot \left(\log t\right)' \Big|_{t=1} = x \cdot \frac{1}{t}\Big|_{t=1} = x$$

$\Rightarrow$

$$e^x = e^{\lim_{n\to\infty} \log\left(\left(1+\frac{x}{n}\right)^n\right)}$$

differentiable $\Downarrow$

$\because e^x$ is continuous

$$\stackrel{\searrow}{=} \lim_{n\to\infty} e^{\log\left(\left(1+\frac{x}{n}\right)^n\right)}$$

$$= \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \qquad \#$$

<u>Recall</u>:

We learned:

$$10^5,\ 2^{\frac{1}{3}},\ \pi^{-\frac{3}{2}},\ e^{-\frac{1}{2}},\ \dots$$

but we didn't define

$$10^{\sqrt{2}},\ 2^{\pi},\ \pi^{-\sqrt{3}},\ e^{-\pi},\ \dots$$

Remark: we define

Previously, we define

$$e^x = \exp(x) = (\log)^{-1}(x) \qquad \forall x \in \mathbb{R}$$

In general, we have

Def (7.5.1)

Let $a > 0$, $x \in \mathbb{R}$. (not necessarily in $\mathbb{Q}$)

Define

$$a^x = e^{\log a^x}$$

$$a^x := \underset{(\log)^{-1}}{\exp}(x \cdot \log a) = e^{x \log a}$$

which is the unique continuous function coinciding with a^x with $x = \frac{p}{q} \in \mathbb{Q}$?

$y = a^x$

Thm (7.5.2)

$\forall x, y \in \mathbb{R}$, $a > 0$. $\qquad p, q \in \mathbb{N}$

(i) $a^{x+y} = a^x \cdot a^y$ $\qquad \frac{p}{q}$

(ii) $a^{x-y} = \dfrac{a^x}{a^y}$

(iii) $(a^x)^y = a^{xy}$

$\Rightarrow a^{\frac{1}{q}} = \sqrt[q]{a}$

and

$a^{-\frac{p}{q}} = \dfrac{1}{\sqrt[q]{a^p}}$

$a^0 = 1$

Pf

(i) $a^{x+y} = e^{(x+y)\cdot \log a} = e^{x\log a + y\log a}$

by the properties of $\exp(x) = e^x$ (last week)

$= e^{x\log a} \cdot e^{y\log a}$

$= a^x \cdot a^y$

(ii) $a^{x-y} = e^{(x-y)\log a} = e^{x\log a}/e^{y\log a}$

$= a^x/a^y$

(iii) $(a^x)^y = (e^{x\log a})^y = e^{xy\log a}$

$= a^{xy}$ #

Cor

$$a^x = \exp(x \cdot \log a) = {}^x\!\sqrt{a}$$

$\forall p \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad x = \frac{p}{q}$

pf

If $x = 1$, then

$$a^1 = \underbrace{\exp(1 \cdot \log a)}_{} \overset{>0}{} = e^{\log a} = a$$

If $x = p \in \mathbb{N}$, then

$$a^p = a^{1+(p-1)} \overset{(i)}{=} a \cdot a^{p-1} = a \cdot a \cdot a^{p-2}$$

$p-1 = 1+p-2$

$$= \cdots = \overbrace{a \cdot a \cdot a \cdots a}^{p \text{ times}}$$

If $x = \frac{1}{q}$, $q \in \mathbb{N}$, then

$$\left(a^{\frac{1}{q}}\right)^q \overset{(iii)}{=} a^{\frac{1}{q} \cdot q} = a^1 = a$$

$a^{\frac{1}{q}} > 0$

$$\Rightarrow \quad a^{\frac{1}{q}} = \sqrt[q]{a}$$

If $x = \frac{p}{q} > 0$, then

$$a^{\frac{p}{q}} = a^{p \cdot \frac{1}{q}} \overset{(iii)}{=} \left(a^p\right)^{\frac{1}{q}}$$

$$= \sqrt[q]{\overbrace{a \cdots a}^{p\text{ times}}} = \sqrt[q]{a^p}$$

$$= \sqrt{a \cdots a} = \sqrt{a}$$

If $x = 0$, then

$$a^0 = \exp(0 \cdot \log a) = e^0 = 1$$

If $x = -\frac{p}{q}$, then

$$1 = a^0 = a^{\frac{p}{q} + (-\frac{p}{q})} \overset{(ii)}{=} a^{\frac{p}{q}} \cdot a^{-\frac{p}{q}}$$

$$\Rightarrow \quad a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{a^p}} \qquad \#$$

<u>Recall</u>

$$\left(x^{\frac{p}{q}}\right)' = \frac{p}{q} \cdot x^{\frac{p}{q} - 1}$$

<u>Thm</u> (7.5.3, 7.5.4)

Let $r \in \mathbb{R}$, $x > 0$.

Then

$$(x^r)' = r \cdot x^{r-1}$$

and for $r \neq -1$,

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$$

pf

$$\frac{d}{dx}(x^r) = \frac{d}{dx}\left(e^{r\log x}\right)$$

$$\overset{\text{chain rule}}{=} e^{r\log x}\cdot\underbrace{(r\log x)'}\ = r\cdot\frac{1}{x}$$

$$= \underbrace{e^{r\log x}}_{\substack{\parallel \\ x^r}}\cdot r\cdot\frac{1}{x}$$

$$= r\cdot x^{r-1} \qquad \#$$

Example

① $(x^\pi)' = \pi\cdot x^{\pi-1}$.

$u = 2x^4+1 \Rightarrow du = 8x^3dx$

② $$\int_0^1 \frac{x^3}{(2x^4+1)^\pi}\,dx = \int_{u(0)=1}^{u(1)=3}\frac{1}{u^\pi}\,\frac{1}{8}\,du$$

$\frac{1}{8}du$

$$= \frac{1}{8}\,\frac{u^{-\pi+1}}{-\pi+1}\Big|_1^3 = \frac{3^{-\pi+1}-1}{8(-\pi+1)} \qquad \#$$

Thm (7.5.5, 7.5.6)

For $a>0$, $f(x)=a^x$ is differentiable on $\mathbb{R}$

and

$$\frac{d}{dx}a^x = a^x \cdot \log a$$

and

$$\int a^x\,dx = \frac{1}{\log a}\cdot a^x + C$$

Pf

$$\frac{d}{dx}a^x = \left(e^{x\log a}\right)' \overset{\text{chain rule}}{=} \underbrace{e^{x\log a}}_{= a^x} \log a$$

$$= a^x \cdot \log a \qquad \#$$

Example

① $\left(3^{x^2}\right)' \underset{y=x^2}{\overset{\text{chain rule}}{=}} \underbrace{\left(3^{x^2}\cdot\log 3\right)}_{\frac{d3^y}{dy}} \cdot \underbrace{(x^2)'}_{\frac{dy}{dx}} = 2x3^{x^2}\cdot\log 3$ #

② $\left(\quad {}^{3x} \right)' \quad \left(\log (x^2+1)^{3x} \right)'$

$$\left((x^2+1)\right)' = \left(e^{\ \ }\right)$$

$$= \left(e^{\overset{y}{\overset{\shortparallel}{3x\cdot\log(x^2+1)}}}\right)'$$

chain

$$= \underbrace{e^{3x\cdot\log(x^2+1)}}_{(x^2+1)^{3x}} \cdot \underbrace{\left(3x\cdot\log(x^2+1)\right)'}_{3\log(x^2+1)+\frac{3x}{x^2+1}\cdot 2x}$$

$$= (x^2+1)^{3x}\cdot\left(3\log(x^2+1)+\frac{6x^2}{x^2+1}\right) \quad \#$$

③ $\displaystyle\int_0^1 x\,5^{-x^2}\,dx$ $\quad u=-x^2 \quad du=-2x\,dx$

$x\,dx = -\frac{1}{2}du$

$$= \int_{u(0)=0}^{u(1)=-1} \overset{\left(\frac{5^u}{\log 5}\right)'}{\overset{\shortparallel}{5^u}}\left(-\tfrac{1}{2}\right)du$$

Recall: $(5^u)' = 5^u\cdot\log 5$

$$-\frac{5^u}{\ \ }\Big|^{-1} = -\frac{\frac{1}{5}}{\ \ }+\frac{1}{\ \ }$$

$$= \quad 2\log 5 \quad \Big|_{u=} 0 \qquad 2\log 5 \cdot 2\log 5$$

$$= \frac{1}{2\log 5} - \frac{1}{10\log 5} \quad \#$$

Recall

Let $a > 0$.

The logarithm with base a,

$$\log_a : (0,\infty) \to \mathbb{R}$$

is the inverse function of

$$\mathbb{R} \to (0,\infty), \quad y \mapsto a^y$$

That is,
$$\log_a(a^y) = y \qquad \forall\, y \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad \forall\, x \in (0,\infty).$$

Thm (7.5.7, 7.5.8)

Let $a > 0$, $a \neq 1$, Then

$$e^{\frac{\log x}{\cancel{\log a}} \cdot \cancel{\log a}} = e^{\log x} = x$$

// def

$$\log_a(a^x) \overset{\text{def}}{=} x \quad \forall x \in \mathbb{R}$$

$$a^{\left(\frac{\log x}{\log a}\right)}$$

since

$$\log_a(x) = \frac{\log x}{\log a} \quad \forall x > 0$$

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{(\log a)\cdot x} \quad \forall x > 0$$

<u>Example</u>

① $\left(\log_5 |x|\right)' = \left(\frac{\log|x|}{\log 5}\right)' = \frac{1}{x \cdot \log 5}$ #

② $\left(\log_2(3x^2+1)\right)' = \left(\frac{\log(3x^2+1)}{\log 2}\right)'$

$$= \frac{1}{\log 2} \quad \frac{6x}{3x^2+1} \quad \#$$

③ $\int \frac{1}{x \log 10}\, dx = \frac{\log|x|}{\log 10} + C$

$$= \log_{10}|x| + C \qquad \#$$

Prop (§11.4) Let $a>0$

(i) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$

a^x is continuous

$$\| $$

$$a^{\lim_{n\to\infty}\frac{1}{n}} = a^0 = 1$$

(ii) $\lim_{n\to\infty} \frac{1}{n^a} = 0$

$$\| $$

$$\lim_{n\to\infty} e^{\overbrace{a\log\frac{1}{n}}^{\to -\infty}} = 0$$

(iii) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

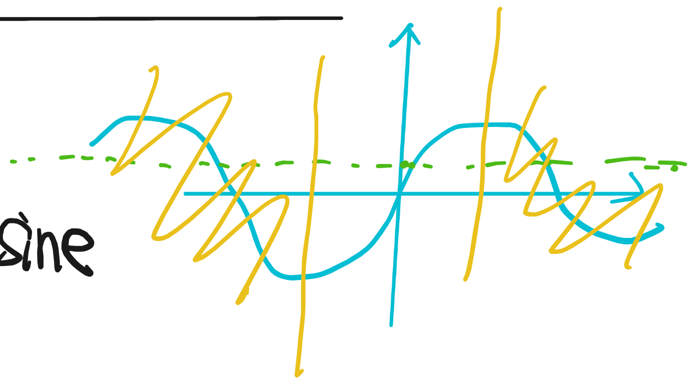
Consider $\lim_{x\to\infty} \frac{\overbrace{\log x}^{\to\infty}}{\underbrace{x}_{\to\infty}} \overset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

(iv) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\log\left(x^{\frac{1}{x}}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\log x}{x}} = e^{0} = 1$$

Inverse trigonometric functions



Def

The restriction of sine function

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

is 1-1, onto, and its inverse is called arc sine, denoted by

$$\sin^{-1} \quad \text{or} \quad \arcsin$$

That is,

$$x = \sin(\sin^{-1} x) \quad \forall x \in [-1, 1]$$

$$y = \sin^{-1}(\sin y) \quad \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

e.g.

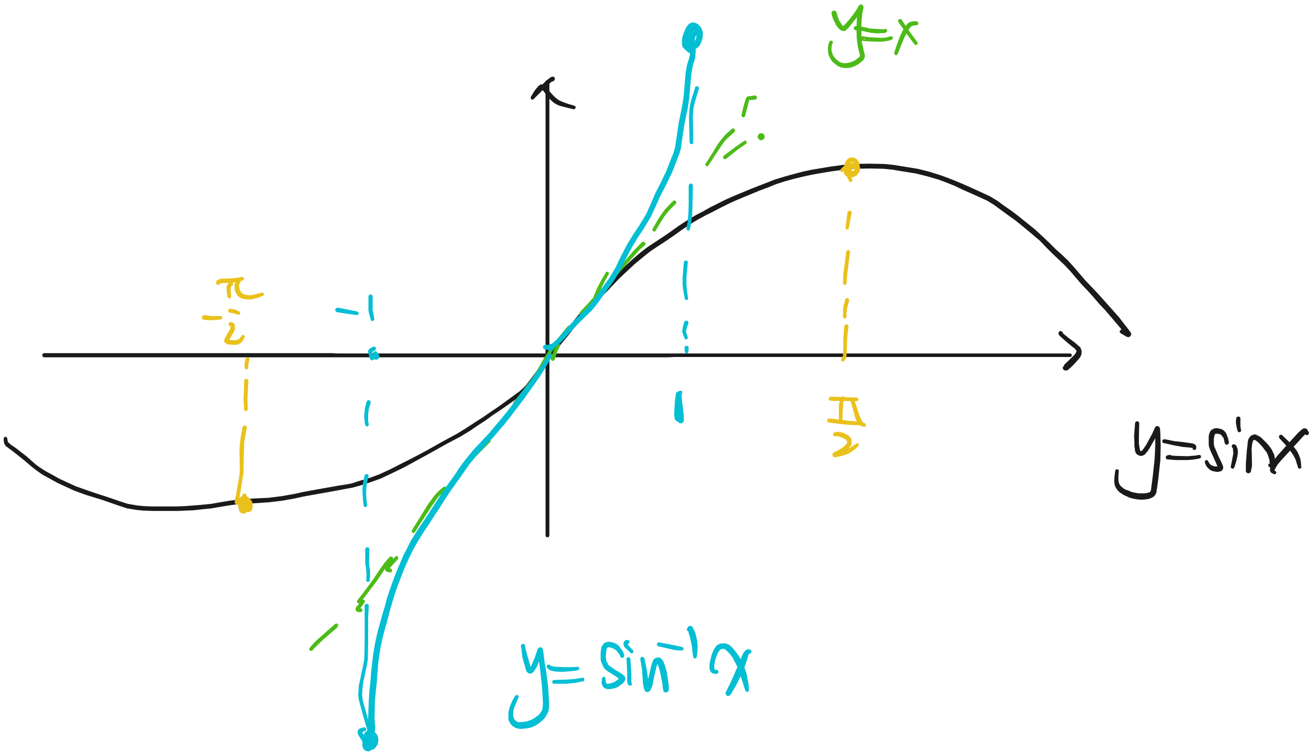
① $\sin\frac{\pi}{6} = \frac{1}{2}$

$\Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

② $\sin^{-1}\left(\sin\left(\frac{13}{6}\pi\right)\right) = \frac{\pi}{6}$

$\neq$

$\frac{13}{6}\pi \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Thm (7.7.3)

$f(x) = \sin^{-1}x$ is differentiable on $(-1, 1)$

and

$$(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

Or equivalently,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

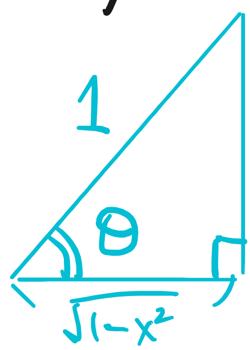
pf

$$\sin(\sin^{-1}x) = x$$

$$\overset{\frac{d}{dx}}{\Rightarrow} \quad 1 = (x)' = \frac{d}{dx}\left(\sin(\sin^{-1}x)\right)$$

$$= \cos(\sin^{-1}x) \cdot \frac{d}{dx}(\sin^{-1}x)$$

$$\Rightarrow (\sin^{-1}x)' = \frac{1}{\cos(\sin^{-1}x)}$$



$\sin\theta = x$

$\Rightarrow \sin^{-1}x = \theta$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

$$= \frac{1}{\sqrt{1-x^2}}$$

#

Example

① $\sin^{-1}(1) = \frac{\pi}{2} \quad (\neq \frac{5}{2}\pi)$

$\sin^{-1}(0) = 0$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

② $\left(\sin^{-1}(3x^2)\right)' = \underbrace{\frac{d}{dy}\sin^{-1}y}_{=\frac{1}{\sqrt{1-y^2}}} \cdot \frac{dy}{dx}$ $\quad y=3x^2$

$= \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x$ #

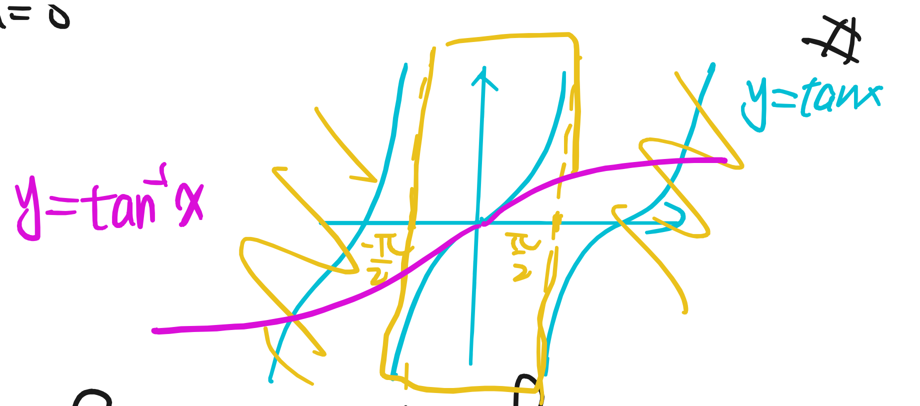
③ $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{\sqrt{4}} \int_0^{\sqrt{3}} \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,\boxed{dx}$ $\quad 2du$

$u = \frac{x}{2} \Rightarrow du = \frac{1}{2}dx$

$= \frac{1}{\cancel{2}} \int_{u(0)=0}^{u(\sqrt{3})=\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\,\cancel{2}\,du$

$= \sin^{-1}u \Big|^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

$u=0$



Def

The restriction of tangent function

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

is 1-1, onto, and its inverse is called arc tangent, denoted by

$$\tan^{-1} \qquad \text{or} \qquad \arctan$$

That is,

$$\tan(\tan^{-1} x) = x \qquad \forall\, x \in \mathbb{R}$$

$$\tan^{-1}(\tan y) = y \qquad \forall\, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

e.g. $\tan^{-1}(1) = \frac{\pi}{4}$, $\tan^{-1}(0) = 0$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Thm (7.7.8)

$f(x) = \tan^{-1} x$ is differentiable on $\mathbb{R}$.

and

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

or

$$\boxed{\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C}$$

pf

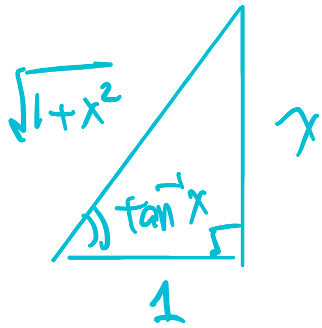
$$x = \tan(\tan^{-1} x)$$

$(\tan y)' = \sec^2 y$

$$\Rightarrow \quad 1 = \frac{d}{dx}\left(\tan(\tan^{-1} x)\right)$$

$$= \sec^2(\tan^{-1} x) \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow \quad (\tan^{-1} x)' = \cos^2(\tan^{-1} x)$$



$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$

$$= \frac{}{1+x^2} \qquad \#$$

Example

① $$\left(\tan^{-1}\left(\frac{x}{3}\right)\right)' = \frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\left(\frac{x}{3}\right)'$$

$$= \frac{1}{1+\frac{x^2}{9}}\cdot\frac{1}{3} = \frac{3}{9+x^2}$$

\#

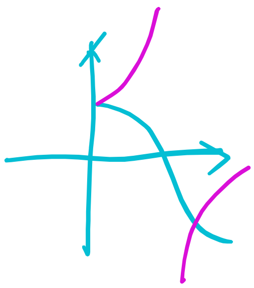
② $$\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{4}\int_0^2 \frac{1}{1+\left(\frac{x}{2}\right)^2}\,\overset{2du}{\boxed{dx}}$$

$u=\frac{x}{2} \Rightarrow du=\frac{1}{2}dx$

$$= \frac{1}{2}\cancel{\frac{1}{4}}\int_{\underset{0}{\underline{u(0)}}}^{\overset{\cancel{\frac{1}{2}}\,1}{\underline{u(1)}}} \frac{1}{1+u^2}\,\cancel{2}du$$

$$= \frac{1}{2}\int_0^{\cancel{\frac{1}{2}}\,1} \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u\Big|_{u=0}^{\cancel{\frac{1}{2}}\,1}$$

$$= \frac{1}{2}\left(\tan^{-1}\cancel{\tfrac{1}{2}}\,1 - \tan^{-1}0\right) = \frac{1}{2}\left(\frac{\pi}{4}-0\right)$$



$$= \frac{\pi}{8} \qquad \#$$

Similarly,

$$\cos^{-1} = \arccos : [-1,1] \to [0,\pi]$$

$\sec^{-1} = \operatorname{arcsec} : (-\infty,-1] \cup [1,\infty) \to [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$

$\cot^{-1} = \operatorname{arccot} : \mathbb{R} \to (0,\pi)$

$\csc^{-1} = \operatorname{acrcsc} : (-\infty,-1] \cup [1,\infty) \to [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$

Thm (7.7.3, 7.7.8, 7.7.12)

$$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\csc^{-1}x)' = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$(\cos^{-1}x)' = \frac{-1}{\sqrt{1-x^2}} \qquad (\sec^{-1}x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(\tan^{-1}x)' = \frac{1}{1+x^2} \qquad (\cot^{-1}x)' = \frac{-1}{1+x^2}$$

$x = \sec(\sec^{-1}x)$

$\Rightarrow 1 = \sec(\sec^{-1}x)\tan(\sec^{-1}x) \cdot (\sec^{-1}(x))'$

$x \qquad \sqrt{x^2-1}$

$\sin(\frac{\pi}{2}-x) = \cos x$

Remark

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$x = \cos^{-1}(\cos x)$

$= \cos^{-1}(\sin(\frac{\pi}{2}-x))$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

$\Downarrow$

$x \in (0, \frac{\pi}{2})$

$\alpha + \theta = \pi - \frac{\pi}{2} = \frac{\pi}{2} \qquad \theta = \sin^{-1}x$

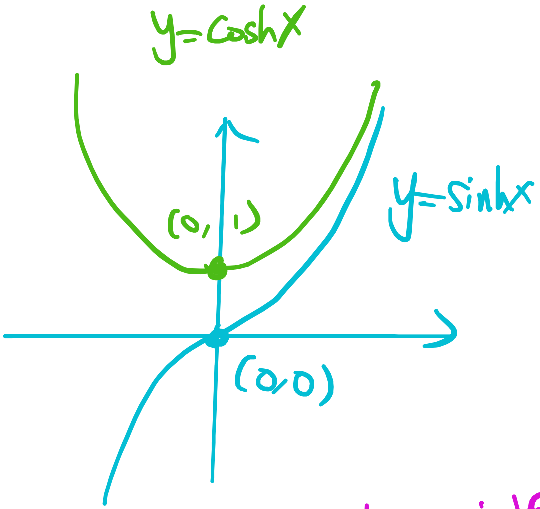
$\alpha = \cos^{-1}x$

雙曲線的

Hyperbolic sine and cosine

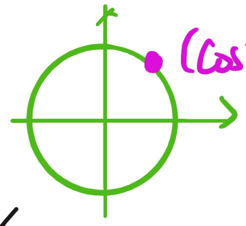
The hyperbolic sine is

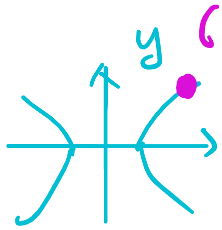
$$\sinh x := \frac{e^x - e^{-x}}{2}$$



The hyperbolic cosine is

$$\cosh x := \frac{e^x + e^{-x}}{2}$$





雙曲線

$x^2 - y^2 = 1$

Remark

- $(\sinh x)' = \cosh x$
- $(\cosh x)' = \sinh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Example

$$\int^{\log 2} x \cosh x \, dx$$

$$\int u(x) v'(x) dx = u(x) v(x) - \int v(x) u'(x) dx$$

$$v(x) = \sinh x \qquad v'(x) = \cosh x$$
$$u(x) = x \qquad u'(x) = 1$$

$$= \underbrace{x \sinh x \Big|_0^{\log 2}} - \int_0^{\log 2} \sinh x \, dx$$

$$\log 2 . \sinh(\log 2) - 0$$
$$\log 2 \cdot \frac{e^{\log 2} - e^{-\log 2}}{2}$$
$$= \log 2 . \frac{2 - \frac{1}{2}}{2} = \frac{3}{4} \log 2$$

$$= \frac{3}{4} \log 2 - \cosh x \Big|_0^{\log 2}$$

$$= \frac{3}{4} \log 2 - \frac{e^{\log 2} + e^{-\log 2}}{2} + \frac{e^0 + e^{-0}}{2}$$

$$= \frac{3}{4} \log 2 - \frac{5}{4} + 1$$

$$= \frac{3}{4} \log 2 - \frac{1}{4} \qquad \#$$

Techniques of integration

Recall

① $\int x^r dx = \begin{cases} \frac{1}{r+1} x^{r+1} + C & r \neq -1 \\ \log|x| + C & r = -1 \end{cases}$

② $\int e^x dx = e^x + C$

③ $\int \sin x \, dx = -\cos x + C$

④ $\int \cos x \, dx = \sin x + C$

⑤ $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

⑥ $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

⑦ $\int f'(u(x)) \cdot u'(x) \, dx = f(u(x)) + C$

⑧ $\int u'(x) \cdot v'(x) \, dx = u(x) \cdot v(x) - \int v(x) \, u'(x) \, dx$

Example

$u(x)=\log x \quad u'(x)=\frac{1}{x}$

$v(x)=\frac{x^2}{2} \quad v'(x)=x$

① $\int x\log x\,dx \overset{\circledS}{=} \frac{x^2}{2}\log x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C$ #

$u(x)=\cos x \quad u'(x)=-\sin x$

$v(x)=e^x \quad v'(x)=e^x$

② $\int e^x\cos x\,dx \overset{\circledS}{=} e^x\cos x + \int e^x(+\sin x)\,dx$

$u(x)=\sin x \quad v'(x)=e^x$

$\overset{\circledS}{=} e^x\cos x + \left(e^x\sin x - \int e^x\cos x\,dx\right)$

$\Rightarrow \int e^x\cos x\,dx = \frac{e^x\cos x + e^x\sin x}{2} + C$

double check :

$\left(\frac{e^x(\cos x + \sin x)}{2}\right)' = \frac{e^x}{2}(\cos x + \cancel{\sin x}) + \frac{e^x}{2}(\cancel{-\sin x} + \cos x)$

$= e^x\cos x$ #

$v' \quad u$

③ $$\int 1\cdot \log x \, dx \overset{\text{Ⓢ}}{=} x\cdot\log x - \int x\cdot \frac{1}{x}\,dx$$

$$= x\cdot\log x - x + C \quad \#$$

④ $$\int \overset{v'}{1}\cdot \overset{u}{\sin^{-1}x}\, dx \overset{\text{Ⓢ}}{=} x\sin^{-1}x - \int \underbrace{x\frac{1}{\sqrt{1-x^2}}}_{=\left(-(1-x^2)^{\frac{1}{2}}\right)'} dx$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C \quad \#$$

⑤ $$\int \overset{v'}{1}\cdot \overset{u}{\tan^{-1}x}\, dx \overset{\text{Ⓢ}}{=} x\tan^{-1}x - \int \underbrace{x\cdot\frac{1}{1+x^2}}_{=\left(\frac{1}{2}\log(1+x^2)\right)'} dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C \quad \#$$

<u>Q:</u>

(*) $$\int \sin^{p}x \cdot \cos^{8}x \, dx = ?$$

<u>Recall</u>

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{ⓐ}$$

$$\ldots \cos\alpha\sin\beta \quad \text{ⓑ}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad ⓑ$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad ⓒ$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad ⓓ$$

$\frac{ⓐ+ⓑ}{2}$:

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$\frac{ⓐ-ⓑ}{2}$:

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$$

$\frac{ⓒ+ⓓ}{2}$:

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

$\frac{ⓓ-ⓒ}{2}$:

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

Example $\quad \sin x\cdot\sin x = \frac{1}{2}\left(\cos(x-x) - \cos(x+x)\right) = \frac{1-\cos 2x}{2}$

① $$\int \underline{\sin^2 x}\,dx = \int \frac{1-\cos 2x}{2}\,dx$$

$$= \int \frac{1}{2}\,dx - \int \frac{1}{2}\cos 2x\,dx \qquad = \left(\frac{\sin 2x}{4}\right)'$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C \quad \#$$

$$\frac{}{2} - \frac{}{4} + C \quad \#$$

② $\int \underline{\sin^2 x \cos^2 x}\, dx$

$$\parallel\ (\sin x \cos x)^2 = \left(\frac{1}{2}\sin 2x\right)^2 = \frac{1}{2}\sin^2(2x)$$

$$= \frac{1}{4} \cdot \frac{1-\cos 4x}{2}$$

$$= \frac{1}{8}\int 1-\underline{\underline{\cos 4x}}\, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C \quad \#$$

$$\parallel\ \left(\frac{\sin 4x}{4}\right)'$$

$p+q$ is even $\Rightarrow$ ⊛ can be computed like ①, ②.

If $p+q$ is odd, then ...

③ $\int \cos^2 x \underbrace{\sin x\, dx}_{-du}$ $u = \cos x$, $du = -\sin x \cdot dx$

$$= -\int u^2\, du = -\frac{u^3}{3} + C$$

$$= -\frac{\cos^3 x}{} + C \quad \#$$

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$1-u^2 = 1-\cos^2 x$

$\sin^2 x \cdot \sin x$ $\quad -du$

$u = \cos x$

④ $\int \cos^2 x \sin^3 x \, dx$

$= -\int u^2 (1-u^2) \, du = \cdots$

1 Problem 1

Suppose $f \in \mathcal{C}^2[0,1]$ such that $f(0) = f(1) = 0$ and $f''(x) < 0$ for all $x \in (0,1)$. Prove that $f(x) > 0$ for all $x \in (0,1)$.

- 證明 $f(0)$ 跟 $f(1)$ 是局部或全局最小無法直接得到 $f(x) > 0$。
- 本題不可以直接說 ”因為凹口向下所以 $f(x) > 0$ ” 。

2 Problem 2

(a) $\lim\limits_{x \to 1} \frac{x^{1/2} - x^{1/4}}{x-1}$. (b) $\lim\limits_{x \to 0} \frac{x + \sin(\pi x)}{x - \sin(\pi x)}$.

- 使用 L’Hospital 法則計算時請檢查（並註明）是不定型。非不定型的極限不可以使用 L’Hospital 法則。

Average	43.654867
Average (except zero)	49.33
Quartile 0	0
Quartile 1	39
Quartile 2	51
Quartile 3	56
Quartile 4	60
標準差	18.825690
非零數	100
不到50%人數	17
滿分人數	14