Calculus, Fall 2025, week 12

f: continuous

$|a+b| \leq |a|+|b|$

Thm (§5.8)

(i) $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

(ii) $$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\cdot u'(x)$$

奇函數

(iii) If f is odd function on $[-a,a]$,

i.e. $f(-x) = -f(x) \quad \forall x \in [-a,a]$,

then

$$\int_{-a}^{a} f(x)\,dx = 0$$

pf

$$= \underbrace{\int_{-a}^{0} f(x)\,dx}_{} + \cancel{\int_0^a f(x)\,dx}$$

$$\int_a^b f(u(x))\cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

$\| \quad u=-x \Rightarrow du = -dx$

$$\int_{u(-a)}^{u(0)} \underbrace{f(-u)}_{-f(u)}(-du) \overset{\text{odd}}{=} \cancel{(-1)} \int_a^0 f(u)\,du$$

$$\cancel{\int}^{a} f(u)\,du$$

$$= \cancel{-\int_0 f(u)\,du}$$

$$= 0$$

#

偶函數

(iv) If f is an even function on $[-a,a]$

i.e. $f(-x) = f(x) \quad \forall x \in [-a,a]$,

then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

(v) If f is continuous on $[a,b]$

and $\displaystyle\int_a^b |f(x)|\,dx = 0$

then $f(x) = 0 \quad \forall x \in [a,b]$.

pf

Assume $f(x_0) \neq 0$ for some $x_0 \in [a,b]$

Assume $x_0 \neq a,b$ (exer: Consider the case $x_0 = a$ or b)

Since $f(x)$ is continuous,

$g(y) = |y|$

$(g \circ f)(x) = |f(x)|$

$|f(x)|$ is also continuous on $[a,b]$

$$\Rightarrow \lim_{x \to x_0} |f(x)| = |f(x_0)| \gneqq 0$$

$a \le x_0 - \delta$, $x_0 + \delta \le b$

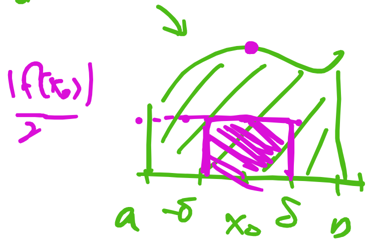
For $\varepsilon = \dfrac{|f(x_0)|}{2} > 0$, $\exists\, \delta > 0$ s.t.

$$\Big|\, |f(x)| - |f(x_0)| \,\Big| < \frac{|f(x_0)|}{2} \quad \text{when } |x - x_0| < \delta$$

$$\rotatebox{90}{$\le$}$$

$$|f(x_0)| - |f(x)|$$

$\int_a^b |f(x)|\,dx$ $\quad y = |f(x)|$



$$\Rightarrow \quad |f(x)| \underset{(*)}{>} \frac{|f(x_0)|}{2} > 0$$

$$\Rightarrow \int_a^b |f(x)|\,dx \ge \int_{x_0-\delta}^{x_0+\delta} |f(x)|\,dx$$

$$\overset{\text{by } (*)}{\ge} \int_{x_0-\delta}^{x_0+\delta} \frac{|f(x_0)|}{2}\,dx$$

$$= \frac{|f(x_0)|}{2} \cdot \underbrace{\left(x_0+\delta - (x_0-\delta)\right)}_{2\delta} = |f(x_0)| \cdot \delta > 0$$

$$(\longrightarrow \longleftarrow)$$

So $f(x) = 0 \quad \forall\ x \in [a,b]$. #

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = 0$$

Example

① $\frac{d}{dx}\left(\int_{x}^{2x} \frac{1}{1+t^2}\,dt \right)$

$= \frac{d}{dx}\left(-\int_{\cancel{x}\ 0}^{\cancel{0}\ x} \frac{1}{1+t^2}\,dt + \int_{0}^{2x} \frac{1}{1+t^2}\,dt \right)$

$\overset{(ii)}{=} -\frac{1}{1+x^2} + \frac{1}{1+(2x)^2} \cdot 2$

Note: $f(x) = a_{2k+1} x^{2k+1} + a_{2k-1} x^{2k-1} + \cdots + a_3 x^3 + a_1 x$ is an odd function: $f(-x) = -f(x)$

(since $(-x)^{2k+1} = (-1)^{2k+1} \cdot x^{2k+1} = -x^{2k+1}$)

② $\int_{-1}^{1} x\,dx$

$\overset{(iii)}{=} 0$

$\int^{1} \sin x\,dx =$ 0

③ $\displaystyle\int_{-1}^{1} x^2\,dx$

$\overset{(iv)}{=} 2\displaystyle\int_{0}^{1} x^2\,dx$

$= 2 \left.\dfrac{x^3}{3}\right|_0^1 = 2\cdot\dfrac{1}{3} = \dfrac{2}{3}$ #

Note:

$g(x) = a_{2k}x^{2k} + a_{2k-2}x^{2k-2} + \cdots + a_2x^2 + a_0$

is an even function

since $(-x)^{2k} = (-1)^{2k}\cdot x^{2k} = x^{2k}$

Remark

For functions with finite discontinuities we can compute their integrals:

Let

$$f(x) = \begin{cases} f_1(x), & x \in [a_0, a_1) \\ f_2(x), & x \in [a_1, a_2) \\ \vdots & \\ f_n(x), & x \in [a_{n-1}, a_n] \end{cases}$$

with $f_i \in C[a_{i-1}, a_i]$.

Then $\int^{a_n} \ldots \quad \int^{a_1} \ldots \quad \int^{a_2} \ldots$

$$\int_{a_0} f(x)\,dx = \int_{a_0} f_1(x)\,dx + \int_{a_1} f_2(x)\,dx + \cdots$$

$$+ \int_{a_{n-1}}^{a_n} f_n(x)\,dx$$

Example

④ If $f(x) = \begin{cases} x, & x > 0 \\ 2, & x = 0 \\ 1, & x < 0 \end{cases}$

then $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} \underbrace{f(x)}_{=1} dx + \int_{0}^{1} \underbrace{f(x)}_{=x} dx$

$= 1 \cdot (0-(-1)) + \left.\frac{x^2}{2}\right|_0^1$

$= 1 + \frac{1}{2} = \frac{3}{2}$ #

⑤ If $g(x) = \begin{cases} 0, & x \text{ is rational 有理數} \\ 1, & x \text{ is irrational 無理數} \end{cases}$

then $g(x)$ is NOT Riemann integrable

超越(數)

§ Transcendental functions

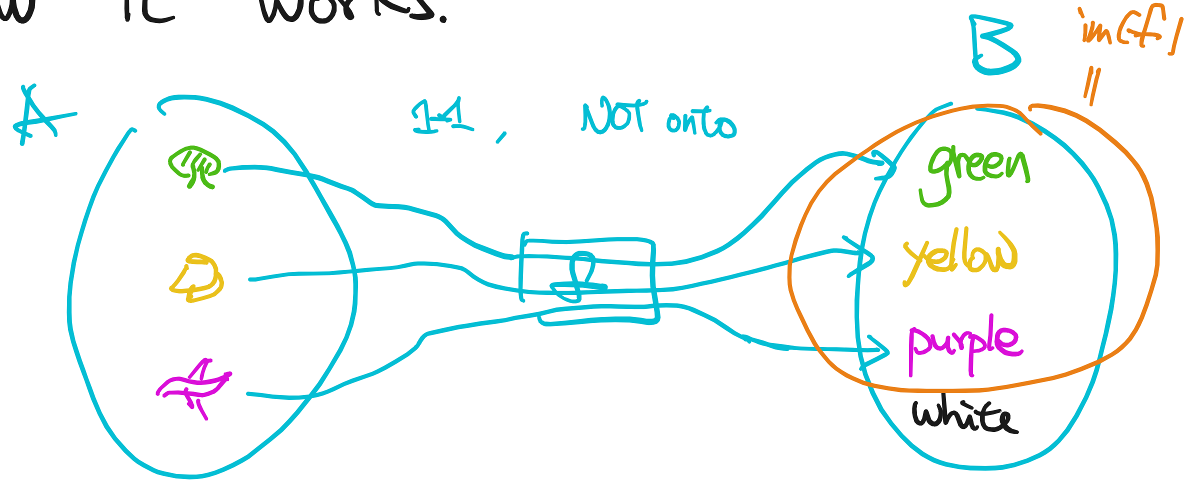
$\cancel{f^2+f=0}$

Inverse functions

Recall a function $f: A \to B$ has 3 ingredients:

- domain $= A =$ set of possible inputs
- codomain $= B =$ set of possible outputs
- how it works.

e.g.



一對一

We say $f: A \to B$ is 1-1 / one-to-one

if $x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

If $x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

值域

The range (or image) of f is the set

$$\mathrm{range}(f) = \mathrm{im}(f) = f(A)$$

$$= \{ f(x) \in B \mid x \in A \}$$

f is onto if $\forall y \in B, \exists x \in A$ s.t. $f(x) = y$

That is, $\mathrm{im}(f) = B$

可逆

A function $f: A \to B$ is invertible if $\exists\ g: B \to A$ s.t.

$$\begin{cases} g(f(x)) = x & \forall x \in A \quad \text{i.e. } g \circ f = id_A \\ f(g(y)) = y & \forall y \in B \quad \text{i.e. } f \circ g = id_B \end{cases}$$

Such a function $g: B \to A$ is called

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the inverse of f and denoted by f^{-1}

the inverse of f, denoted by f^{-1}

Thm

A function $f: A \to B$ is invertible if and only if f is 1-1 and onto

Remark

If $f: A \to B$ is 1-1, then its "corestriction" $f: A \to im(f)$ is 1-1 and onto, hence invertible

(see e.g. Thm 7.1.2)

Example

$x \neq y \Rightarrow x^3 \neq y^3$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $x^3 = y$ $(x = \sqrt[3]{y})$

① $f(x) = x^3$ is 1-1 and onto

$f: \mathbb{R} \to \mathbb{R}$

Since

$$y = f(f^{-1}(y)) = (f^{-1}(y))^3$$

we have

$f^{-1}(y) = \sqrt[3]{y}$ #

② $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$,

is NOT 1-1

since

$-1 \neq 1$, but $g(-1) = (-1)^2 = 1^2$
$= g(1)$

NOT onto

since for $-1 \in \mathbb{R}$.

$\nexists\, x \in \mathbb{R}$ s.t.

$g(x) = -1 < 0$

$0 \leq x^2$

In particular, g is NOT invertible

$x^2 = g(x) = y$ does NOT have a unique sol.

③ Show that $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^5 + 2x^3 + 3x - 4$$

is 1-1

pf

method I Check $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

This is too complicated for this f.

method II Check if f is strictly increasing/ strictly decreasing:

Since

$$f'(x) = \underbrace{5x^4+6x^2}_{\geq 0} + 3 \underset{\neq}{>} 0 \quad \forall x \in \mathbb{R}.$$

the function f is strictly increasing

That is, $\forall x_1, x_2 \in \mathbb{R}$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

So if $x_1 \neq x_2$, then

$$x_1 < x_2 \Rightarrow f(x_1) \underset{\neq}{<} f(x_2) \Rightarrow f(x_1) \neq f(x_2)$$

or

$$x_1 > x_2 \Rightarrow f(x_1) \underset{\neq}{>} f(x_2) \Rightarrow f(x_1) \neq f(x_2)$$

So f is 1-1. #

Thm (Th 7.18)

Thm (Thm 7.1.6)

Let f be 1-1 $\Rightarrow$ a corestriction is invertible, differentiable, $f(a)=b$

If $f'(a)\neq 0$, then f^{-1} is differentiable at b, and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

pf

We skip the proof of the differentiability of f^{-1}!

Since

$$f^{-1}(f(x)) = x$$

$$\Rightarrow \quad \frac{d}{dx}\left(f^{-1}(f(x))\right) = \frac{d}{dx}(x) = 1$$

chain rule!!

$$(f^{-1})'(f(x)) \cdot f'(x)$$

At $x=a$, we have ① by

※ assumption

$$(f^{-1})'\big(\overset{b}{\overset{\shortparallel}{f(a)}}\big) \cdot f'(a) = 1$$

$$\Rightarrow \quad (f^{-1})'(b) = \frac{1}{f'(a)} \qquad \#$$

Example

Let $f(x) = x^3 + \frac{1}{2}x$.

Q: $(f^{-1})'(9) = ?$

Sol

Since $f'(x) = 3x^2 + \frac{1}{2} > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ f is strictly increasing

$\Rightarrow$ f is 1-1

Also, $f(2) = 9$

$\Rightarrow$

$$(f^{-1})'(9) = \frac{}{f'(2)} = \frac{1}{3\cdot 4 + \frac{1}{2}}$$

$$= \frac{2}{25} \qquad \#$$

The logarithm function

Recall that

$$\left(\frac{x^{n+1}}{n+1}\right)' = x^n \quad \forall n \neq -1$$

$$\Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\forall n \neq -1$$

Q: What if $n = -1$?

$$\int \frac{1}{x}\,dx = ?$$

$(\log x)' = \frac{1}{x}$

by fundamental thm of calculus

continuous on $(0, +\infty)$

Def (§7.2)

The function

$\cap x$

$\ln x \ (= \log x) := \int_1^x \frac{1}{t}\,dt$, This integral is OK

for $x>0$, is called the

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(natural) logarithm function

Thm (§7.3)

(i) $\log x$ is differentiable on $(0,\infty)$

(ii) $\frac{d}{dx}(\log|x|) = \frac{1}{x} \quad \forall x \neq 0$

($\Rightarrow \log x$ is strictly increasing on $(0,\infty)$)

pf

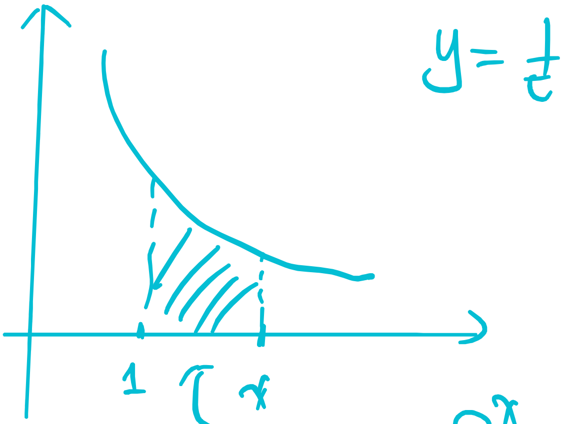
$$\log|x| = \begin{cases} \log x & x>0 \\ \log(-x) & x<0 \end{cases}$$

$$\Rightarrow \frac{d}{dx}(\log|x|) = \begin{cases} \frac{d}{dx}(\log x) = \frac{1}{x}, & x>0 \\ \frac{d}{dx}\log(-x) = \frac{1}{-x}\underbrace{(-x)'}_{=-1} & x<0 \end{cases}$$

$$= \frac{1}{x}$$

$$= \frac{1}{x} \qquad \forall x \neq 0.$$ #

(iii) $\underset{\displaystyle \int_1^x \frac{1}{t}\,dt}{\overset{\displaystyle \log x}{\shortparallel}} \begin{cases} >0 & x>1 \\ =0 \ \left(=\int_1^1 \frac{1}{t}\,dt\right) & x=1, \\ <0 & 0<x<1 \end{cases}$



$$\log x = \int_1^x \frac{1}{t}\,dx$$

Thm (§7.2)

For $a, b > 0$,

(i) $\log(ab) = \log a + \log b$

(ii) $\log\left(\frac{1}{b}\right) = -\log b$

(iii) $\log\left(\frac{a}{b}\right) = \log a - \log b$

(iv) $\log\left(a^{\frac{p}{q}}\right) = \frac{p}{q} \cdot \log a$

pf

Since

$$\frac{d}{dx} \log(x \cdot b) \overset{\text{chain rule}}{=} \frac{1}{xb} \cdot b = \frac{1}{x}$$

$$= \frac{d}{dx} \log x \qquad \forall x > 0$$

there is $C \in \mathbb{R}$ s.t.

$$\log(x \cdot b) = \log x + C \qquad \forall x > 0$$

At $x = 1$,

$$\log b = \underbrace{\log 1}_{=0} + C = C$$

$$\Rightarrow \quad \log(xb) = \log x + \log b \qquad \forall x > 0$$

or

$$\log(ab) = \log a + \log b \quad \Rightarrow (i)$$

#

(ii) $$\log\left(b \cdot \frac{1}{b}\right) = \log(1) = 0$$

$\| (i)$

$$\log b + \log \frac{1}{b}$$

$$\Rightarrow \log \frac{1}{b} = -\log b \qquad \#$$

(iii) $$\log \frac{a}{b} = \log\left(a \cdot \frac{1}{b}\right) \overset{(i)}{=} \log a + \log \frac{1}{b}$$

$$\overset{(ii)}{=} \log a - \log b \qquad \#$$

$a \cdot a^{p-2}$

$\|$

Step1

(iv) $$\log(a^p) = \log(a \cdot a^{p-1}) = \log a + \log a^{p-1}$$

$$= \log a + \log a + \log a^{p-2} = \cdots$$

p times

$$= \log a + \log a + \cdots + \log a$$

$$= p \cdot \log a .$$

Step2

$$\log\left(\left(a^{\frac{1}{q}}\right)^{q}\right) = \log a$$

$$\| \text{ step1}$$

$$q \cdot \log\left(a^{\frac{1}{q}}\right) \quad \Rightarrow \quad \log\left(a^{\frac{1}{q}}\right) = \frac{1}{q} \cdot \log a$$

Step3

$$\log\left(a^{\frac{p}{q}}\right) = \log\left(\left(a^{\frac{1}{q}}\right)^{p}\right)$$

$$\underset{\text{Step1}}{=} p \cdot \log\left(a^{\frac{1}{q}}\right) \underset{\text{Step2}}{=} \frac{p}{q} \cdot \log a$$

#

Remark

In particular,

$\log 2 > 0$ can be arbitrary large

and

$$\log 2^{p} = \underbrace{p \cdot \log 2}_{} \quad \forall p \in \mathbb{N}$$

$\to +\infty$ as $p \to \infty$

Also,

$$\log \frac{1}{2^{p}} = -p \log 2 \to -\infty$$

as $p \to \infty$

So $\forall y \in \mathbb{R}$,

<u>case 1</u> $y = 0$,

$\Rightarrow \log 1 = 0 = y$

<u>case 2</u> $y > 0$

$\Rightarrow \exists p$ s.t. $\overset{\log(2^p)}{\overset{\parallel}{p \log 2}} > y$

Since $\log x$ is continuous on $(0, +\infty)$, by the intermediate value thm, $\exists x \in (0, 2^p)$ s.t.

$$\log x = y$$

<u>case 3</u> $y < 0$, $\log(2^{-p})$ $\quad 0$

$\parallel \qquad \parallel$

$\Rightarrow \exists\, p$ s.t. $\underbrace{-p \log 2} < y < \log(1)$

$\Rightarrow$ by IVT, $\exists\, x \in (2^{-p}, \cancel{0}\ 1)$ s.t.

$$\log x = y.$$

<u>Conclusion</u>

$$\operatorname{im}(\log) = \mathbb{R}$$

$\Rightarrow$ $\log : (0, +\infty) \to (-\infty, \infty)$

is 1-1 and onto $\Rightarrow$ invertible

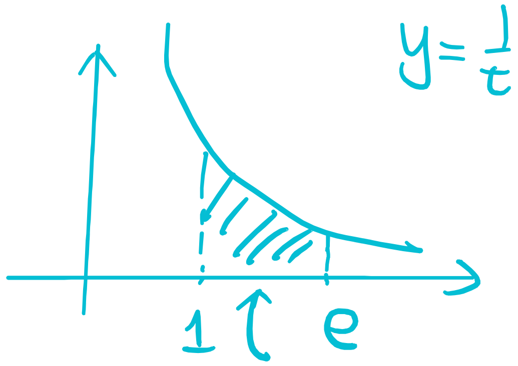
$\Rightarrow \exists!\ \underset{\sim}{e} \overset{= (\log)^{-1}(1)}{} \in (0, \infty)$ s.t.

$$\log(e) = 1.$$

<u>Def</u> (§7.2)

The number e is the unique number s.t.

$$\log(e) = \int_1^e \frac{1}{t}\,dt = 1$$



area = 1

NOT a root of any polynomial
= with integer coeff

超越數

Remark

In fact, $e = 2.718\cdots$ is a transcendental number, and in particular, $e \notin \mathbb{Q}$

x	1	2	e	3	4
$\log x$	0	0.69...	1	1.1...	1.39...

Remark

Since $(\log|x|)' = \frac{1}{x}$ $\forall x \neq 0$,

we have

$$\int \frac{1}{x}\,dx = \log|x| + C.$$

Example

① $\displaystyle\int_1^2 \overbrace{\frac{6x^2+2}{\underbrace{x^3+x+1}_{u=}}\,dx}^{2\,du}$ Let $u = x^3+x+1$

$du = (3x^2+1)\,dx$

$$= \int_{\underbrace{u(1)}_{=3}}^{\overbrace{u(2)}^{=11}} \frac{2}{u}\,du$$

$$= 2\log|u|\,\Big|_{u=3}^{11} = 2(\log 11 - \log 3)$$

$$= 2\log\frac{11}{3} \quad \#$$

② $\displaystyle\int \tan x\,dx = \int \overbrace{\frac{\sin x}{\underbrace{\cos x}_{u}}\,dx}^{-du}$ $u = \cos x$

$du = -\sin x\,dx$

$$= -\int \frac{1}{u}\,du = -\log|u| + C$$

$$= -\log|\cos x| + C$$

$$= \log|\sec x| + C$$

③ Similarly,

$$\int \cot x \, dx = \log|\sin x| + C$$

$$\int \sec x \, dx = \log|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \log|\csc x - \cot x| + C$$

NOTE: If the interval of integration has a point where "$\log 0$" appears, then the formula might be wrong. (NOT integrable)

④ $\int_0^{\pi} \underbrace{\tan x}_{= \frac{\sin x}{\cos x}} \, dx$ does NOT exist

$= \dfrac{\sin x}{\cos x}$ is discontinuous at $x = \dfrac{\pi}{2}$

$$\log|\sec x|\Big|_{x=\frac{\pi}{2}} = \cancel{\log\left|\frac{1}{0}\right|}$$

$$\neq \log|\sec x|\Big|_0^{\pi}$$

Exponential function

Recall

$$\log : (0,\infty) \to (-\infty,\infty)$$

is invertible

Def

The (natural) exponential function

$$\exp : (-\infty,\infty) \to (0,\infty)$$

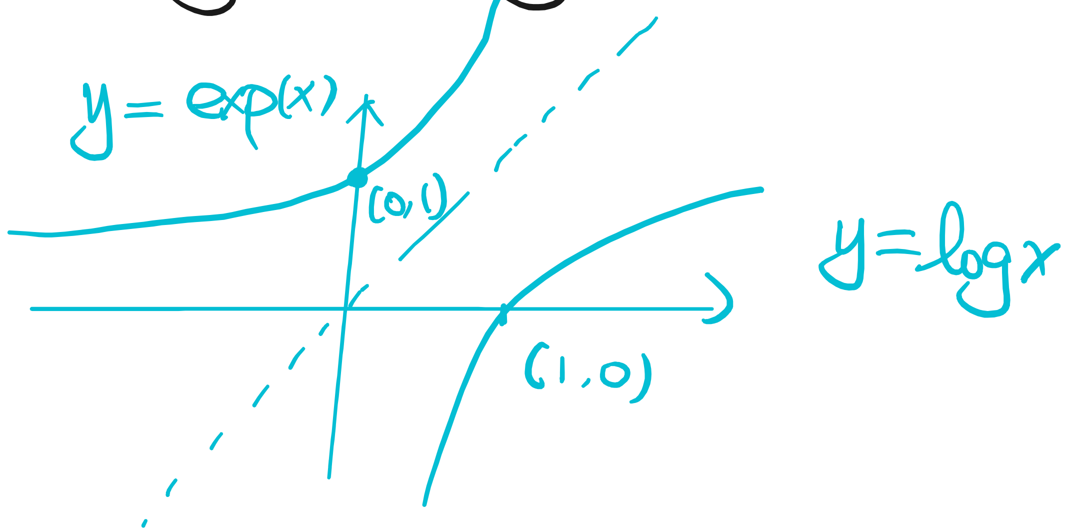
is the inverse function of

$$\log : (0,\infty) \to (-\infty,\infty)$$

That is,

$$\log(\exp(x)) = x \qquad \forall x \in \mathbb{R}$$

$$\exp(\log y) = y \qquad \forall y > 0.$$



Thm (§7.4)

because $\log(1) = 0$, $\log(e) = 1$

(i) $\exp(0) = 1$, $\exp(1) = e$

(ii) $\exp(x) > 0 \quad \forall x \in \mathbb{R}$

(iii) $\displaystyle\lim_{x \to -\infty} \exp(x) = 0$

$$\lim_{x \to -\infty} \log(\exp x) = \lim_{x \to -\infty} x \to -\infty$$

(iv) exp is strictly increasing

$\because$ log is strictly increasing

$$\exp(x) < \exp(y) \Leftrightarrow \log(\exp(x)) < \log(\exp(y))$$

$$\log(\exp(x)) = x, \quad \log(\exp(y)) = y$$

(v) $\exp(a+b) = \exp(a)\cdot\exp(b)$

$$\log(\exp(a+b)) = a+b$$

$$= \log(\exp a) + \log(\exp b)$$

$$= \log(\exp(a)\cdot\exp(b))$$

log is 1-1

$$\Rightarrow \quad \exp(a+b) = \exp(a)\cdot\exp(b) \quad \#$$

$n \in \mathbb{N}$

(vi) $\exp(n\cdot a) = \exp(\overbrace{1+1+\cdots+1}^{n \text{ times}})$

$$(\exp(a))^n \overset{(v)}{=} \underbrace{\exp(1 \cdot a)}_{= e} \cdot \exp(\overbrace{1+1+\cdots+1}^{n-1 \text{ times}}) \qquad a+\cdots+a$$

$$= \cdots = \overbrace{e\cdot e \cdots e}^{n \text{ times}} = e^n$$

(vii) $\exp(\frac{1}{n}) = e^{\frac{1}{n}}$

$$\exp(n\cdot\tfrac{1}{n}) = \exp(1) = e$$

$$\exp(\tfrac{1}{n} \cdot n) = \exp(1) = e$$

$$\| \text{(vi)}$$

$$\left(\exp(\tfrac{1}{n})\right)^n \quad \Rightarrow \quad \exp(\tfrac{1}{n}) = e^{\frac{1}{n}} \quad \#$$

(viii) $$\exp(\tfrac{p}{q}) = e^{\frac{p}{q}}$$

$$\| \qquad p \text{ times}$$

$$\exp(\underbrace{\tfrac{1}{q} + \cdots + \tfrac{1}{q}}_{p\text{ times}}) = \left(\exp(\tfrac{1}{q})\right)^p = \left(e^{\frac{1}{q}}\right)^p = e^{\frac{p}{q}} \quad \#$$

Summary

$\exp : \mathbb{R} \to (0, \infty)$

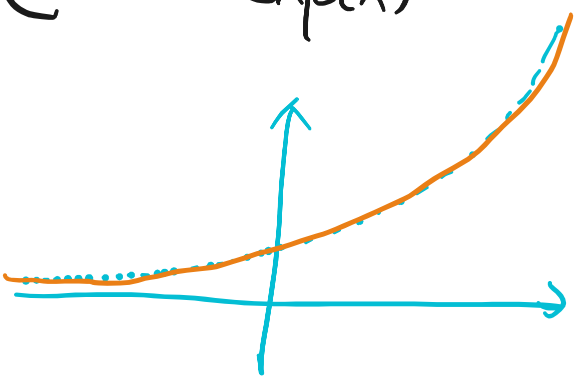
is the continuous function s.t.

$$\exp(x) = e^x \qquad \forall x \in \mathbb{Q}$$

For $x \notin \mathbb{Q}$, e^x is NOT defined yet.

We define

$$e^x = \exp(x) \qquad \text{for } x \notin \mathbb{Q}.$$



$y = e^x,\ x \in \mathbb{Q}$

For $x \in \mathbb{R}$, $\exists \frac{p_n}{q_n} \in \mathbb{Q}$ s.t. $\lim_{n\to\infty} \frac{p_n}{q_n} = x$

$\because$ exp is continuous

$$\exp(x) = \lim_{n\to\infty} \exp\left(\frac{p_n}{q_n}\right) = \lim_{n\to\infty} e^{\frac{p_n}{q_n}}$$

So $\exp(x)$ is uniquely determined by $e^{\frac{p}{q}}$

In other words, $\exp(x)$ is the unique continuous function s.t.

$$\exp(x) = e^x \quad \forall x \in \mathbb{Q}$$

<u>Thm</u> (Thm 7.4.9)

$\exp(x) = e^x$ is differentiable on $(-\infty, \infty)$

and

$$\frac{d}{dx} e^x = e^x$$

$$\Rightarrow \int e^x \, dx = e^x + C$$

pf

$$x = \log(e^x)$$

$\Rightarrow$

$$1 = \frac{d}{dx}x = \frac{d}{dx}\left(\log\left(e^{x}\right)\right)$$

$$\overset{\text{chain rule}}{=} \frac{1}{e^{x}} \cdot \frac{d}{dx}e^{x}$$

$$\Rightarrow \frac{d}{dx}e^{x} = e^{x} \quad \#$$

Example

① $$\left(e^{ax}\right)' = e^{ax} \cdot (ax)' = ae^{ax}.$$

② $$\left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$

③ $$\left(x e^{-x}\right)' = (x)' \cdot e^{-x} + x \cdot \left(e^{-x}\right)'$$

$$= e^{-x} - x e^{-x} \quad \#$$