Calculus, Fall 2025, week 11

<u>Thm</u> (L'Hôpital's rule, ∞/∞ type, §11.6)
Let f, g be differentiable on (a,b) and $g'(x) \neq 0$ $\forall x \in (a,b)$. Assume

$$\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty$$

$$\lim_{x\to a^+} g(x) = \infty \text{ or } -\infty$$

and

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \quad \text{exists.}$$

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

$$\text{“} \lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \text{”}$$

(The case $x \to b^-$ is similar.)

<u>Example</u>

$$\cancel{\sec x} \cdot \tan x$$
$$\|$$
$$(\sec x)'$$

$$\lim \frac{\sec x}{\ldots} \quad \sec x \to +\infty \qquad \overset{\text{LHopital}}{=} \quad \lim \ldots$$

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\qquad}{\underbrace{1+\tan x}_{\to +\infty}} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\qquad}{(1+\tan x)'}$$

$$(1+\tan x)' = \sec^{\cancel{2}} x$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}} = \lim_{x\to(\frac{\pi}{2})^-} \sin x = 1 \quad \#$$

<u>Lemma</u>

Let $A, B \in \mathbb{R}$, and $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be seq. s.t. $b_n \to \pm\infty$. Then

$\Rightarrow \frac{B}{b_n} \to 0$

$$\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty} \text{ converges}$$

$$\Leftrightarrow$$

$$\left(\frac{a_n+A}{b_n+B}\right)_{n=1}^{\infty} \text{ converges}$$

Furthermore,

$$\overline{\lim_{n\to\infty}} \frac{a_n}{b_n} = \overline{\lim_{n\to\infty}} \frac{a_n+A}{b_n+B}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n + A}{b_n + B}$$

$$\boxed{\frac{a_n + A}{b_n + B}} = \frac{\boxed{\frac{a_n}{b_n}} + \frac{A}{b_n}}{1 + \frac{B}{b_n}}$$

$\frac{A}{b_n} \to 0$

$\frac{B}{b_n} \to 0$

pf of L'Hôpital's rule

Claim: $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$

It suffices to show that $\forall\, x_n \in (a, b)$ with $x_n \to a$, we have $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L$

$\forall \varepsilon > 0$, since $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\exists\, \delta_\varepsilon > 0$ s.t.

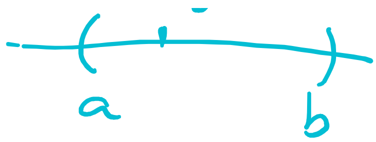
$b-a$

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$$

$f(\infty)$ $\cancel{f(a)}$ $f(x_\varepsilon)$ ✓

x_ε

when $0 < x - a < \delta_\varepsilon$



Let $x_\varepsilon = a + \frac{\delta_\varepsilon}{2} \Rightarrow 0 < x_\varepsilon - a < \delta_\varepsilon$

$$\Rightarrow \left| \frac{f'(x_\varepsilon)}{g'(x_\varepsilon)} - L \right| < \varepsilon.$$

for $\varepsilon = x_\varepsilon - a = \frac{\delta_\varepsilon}{2} > 0$

Since $\lim_{n\to\infty} x_n = a$, $\exists N_\varepsilon$ s.t.

$$a < x_n < x_\varepsilon \qquad \forall\ n \geq N_\varepsilon$$

$$a < x_n < a + (x_\varepsilon - a) = x_\varepsilon$$

So, for $n \geq N_\varepsilon$, since

(a, b)
$\cup$

f, g are differentiable on (x_n, x_ε)

f, g are continuous on $[x_n, x_\varepsilon]$

$\cap$
(a, b)

by Cauchy's MVT, $\exists\ c \in (x_n, x_\varepsilon) \subseteq (a, a+\delta_\varepsilon)$ s.t.

$\Rightarrow 0 < c - a < \delta_\varepsilon$

$$\frac{f(x_\varepsilon) - f(x_n)}{g(x_\varepsilon) - g(x_n)} = \frac{f'(c)}{g'(c)}$$

$$\left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon$$

$$\Rightarrow \left| \frac{f(x_\varepsilon) - f(x_n)}{g(x_\varepsilon) - g(x_n)} - L \right| < \varepsilon$$

$\forall n \geq N_\varepsilon$

$$\Rightarrow \overline{\lim_{n\to\infty}} \left| \frac{f(x_\varepsilon) - f(x_n)}{g(x_\varepsilon) - g(x_n)} - L \right| \leq \varepsilon$$

$$\|$$

$$\left| \overline{\lim_{n\to\infty}} \frac{\overset{A}{\overset{\|}{f(x_\varepsilon)}} \overset{\to \pm\infty}{- f(x_n)}}{\underset{\underset{B}{\|}}{g(x_\varepsilon)} \underset{\to \pm\infty}{- g(x_n)}} - L \right|$$

$A, B \in \mathbb{R}$

$\|$ by Lemma

$$0 \leq \left| \overline{\lim_{n\to\infty}} \frac{f(x_n)}{g(x_n)} - L \right| \leq \varepsilon$$

$\forall \varepsilon > \ldots$

$\forall \varepsilon > 0$

$$\Rightarrow \overline{\lim_{n\to\infty}} \frac{f(x_n)}{g(x_n)} = L$$

Similarly,

$$\underline{\lim_{n\to\infty}} \frac{f(x_n)}{g(x_n)} = L$$

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L \quad \forall (x_n)_{n=1}^{\infty},\ x_n \in (a,b)$$

$$x_n \to a$$

$$\Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L. \quad \#$$

Example

"$0\cdot\infty$"

① $\displaystyle\lim_{x\to(\frac{\pi}{2})^-} \underbrace{\left(\frac{\pi}{2}-x\right)}_{\to 0} \cdot \underbrace{\tan x}_{\to +\infty}$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\overbrace{\left(\frac{\pi}{2}-x\right)\sin x}^{\to 0}}{\underbrace{\cos x}_{\to 0}}$$

$$\frac{1}{-\dfrac{\cos x - \cos\frac{\pi}{2}}{x-\frac{\pi}{2}}} \to (\cos x)'$$

L'Hôpital

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{\underbrace{(-1)\cdot \sin x}_{\to -1} + \underbrace{(\frac{\pi}{2} - x)\cdot \cos x}_{\to 0}}{\underbrace{-\sin x}_{\to -1}}$$

$$= \frac{-1}{-1} = 1 \quad . \#$$

② $\lim_{x \to 0} \left(\underbrace{\frac{1}{\sin x}}_{\to \pm\infty} - \underbrace{\frac{1}{x}}_{\to \pm\infty} \right)$ "$\infty - \infty$"

$$= \lim_{x \to 0} \frac{\overbrace{x - \sin x}^{\to 0}}{\underbrace{x \cdot \sin x}_{\to 0}}$$

L'Hôpital

$$= \lim_{x \to 0} \frac{\overbrace{1 - \cos x}^{\to 0}}{\underbrace{1 \cdot \sin x + x \cdot \cos x}_{\to 0}}$$

L'Hôpital

$$= \lim_{x \to 0} \frac{\overbrace{\sin x}^{\to 0}}{\cos x + 1 \cdot \cos x + x \cdot (-\sin x)}$$

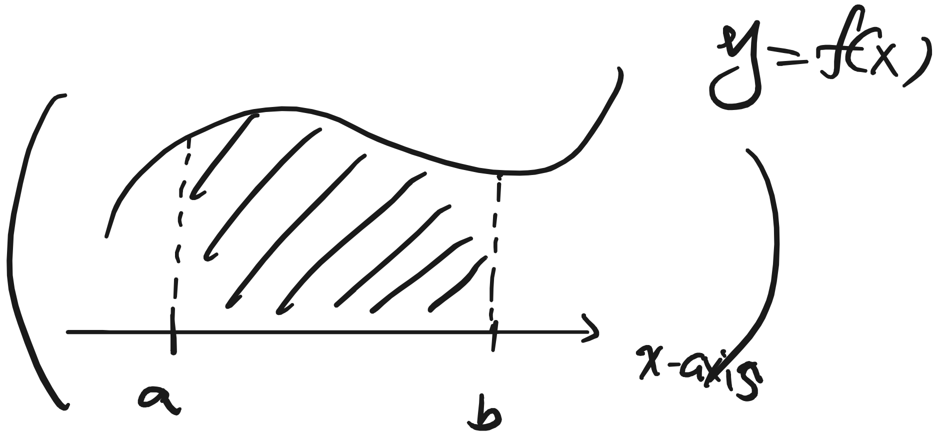
$$1 + 1 + 0 = 2$$

$= \frac{0}{2} = 0$. #

Integration 積分

Let $f(x)$ be a function s.t. $f(x) \geq 0$ $\forall x \in [a,b]$

The integral (積分) of f from a to b, denoted by $\int_a^b f(x)\,dx$, is

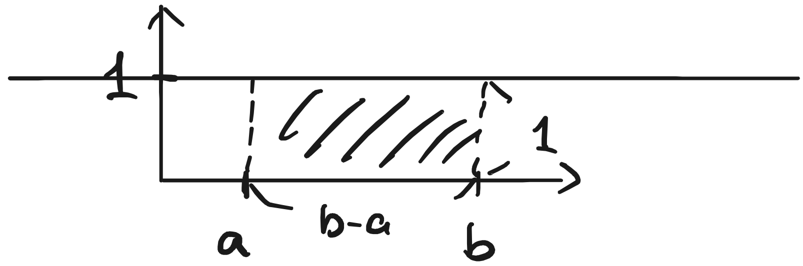
area 

$$= \int_a^b f(x)\,dx.$$

Example

① $f(x) = 1$,

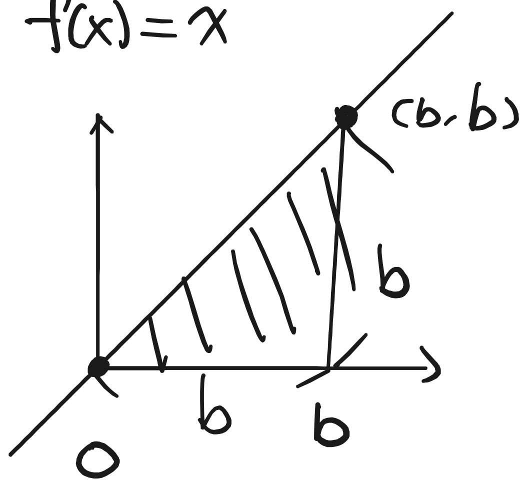
$y = f(x) = 1$



$$\Rightarrow \int_a^b f(x)\,dx = (b-a)\cdot 1 = b-a. \quad \#$$

② $f(x) = x$

$y = f(x) = x$



$$\Rightarrow \int_0^b x\,dx$$

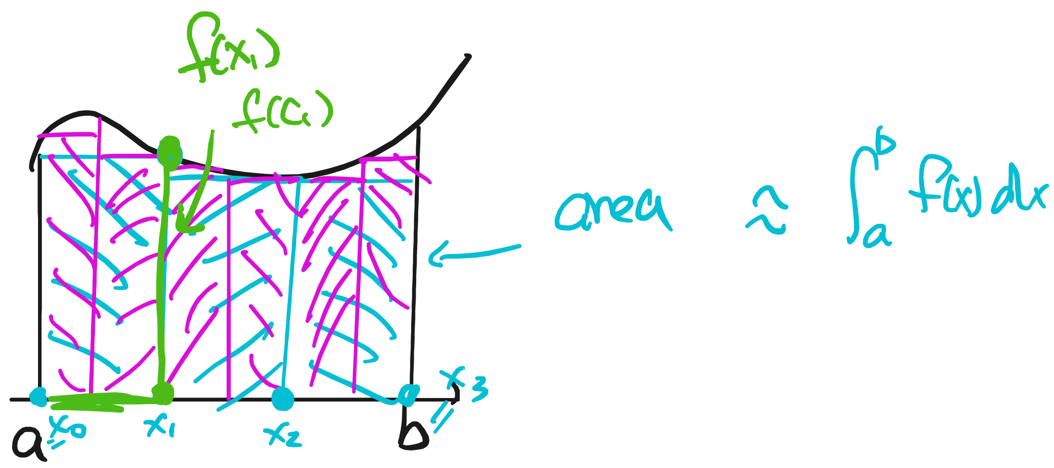
$$= \frac{1}{2}\cdot b\cdot b$$

\#

Q: In general, it is very difficult to compute the area

area() = ?

Idea: find an approximation of $\int$



<u>Def</u> (5.2.1)

A <u>partition</u> P of $[a,b]$ is

a finite seq.

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

The number

$$\|P\| := \max\{|x_1 - x_0|, |x_2 - x_1|, \dots, \dots |x_n - x_{n-1}|\}$$

is called the norm of $\mathcal{P}$

Example

$$\mathcal{P} = \left\{0, \frac{1}{2}, \frac{2}{3}, 1\right\}$$

is a partition of $[0, 1]$,

$$\|\mathcal{P}\| = \max\left\{ \underset{\frac{1}{2}}{\underset{\parallel}{\left|\frac{1}{2} - 0\right|}}, \underset{\frac{1}{6}}{\underset{\parallel}{\left|\frac{2}{3} - \frac{1}{2}\right|}}, \underset{\frac{1}{3}}{\underset{\parallel}{\left|1 - \frac{2}{3}\right|}} \right\}$$

$$= \frac{1}{2}$$

Def (5.2.6)

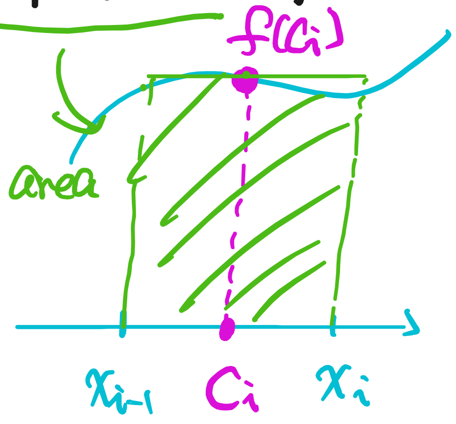
A Riemann sum for f on $[a, b]$

with respect to $\mathcal{P}$ is

d ↔ difference

$$R_f(\mathcal{P}) = \sum_{i=1}^{n} f(c_i) \cdot \overbrace{|x_i - x_{i-1}|}^{\text{"}\Delta x\text{"}}$$

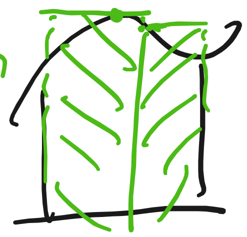
where $c_i \in [x_{i-1}, x_i]$



Remark

Theoretically, people consider

- upper Riemann sum



- lower Riemann sum

google

Def (5.2.7)

可積分

A function f is (Riemann) integrable on $[a,b]$ if the limit

$$\lim_{\|P\|\to 0} R_f(P)$$

exists and does not depend on the choices of Riemann sums.

In this case, the number

$$\int_a^b f(x)\,dx := \lim_{\|P\|\to 0} R_f(P)$$

is called the (definite) integral (定)積分 of f from a to b

Thm

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Remark

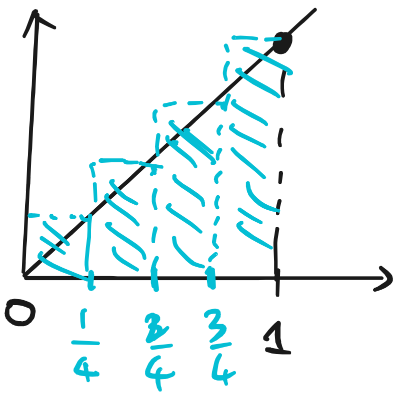
The values of f at finite points do not affect $\int_a^b f(x)\,dx$

Example

$f(x) = x$, $\quad x \in [0,1]$

$$P_n = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \cdots, \tfrac{n}{n}\}$$

$$C_i = \frac{i}{n} \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$$



$$R_f(P_n) = \sum_{i=1}^{n} \underbrace{f(c_i)}_{} \cdot \left(\frac{i}{n} - \frac{i-1}{n}\right), \quad f(c_i) = c_i$$

$$= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} i$$

$$= \frac{1}{n^2} \cdot \frac{(1+n)n}{2} = \frac{1+n}{2n}$$

$$\longrightarrow \frac{1}{2} \quad \text{as } n \to \infty$$

So $\displaystyle\lim_{n\to\infty} R_f(P_n) = \frac{1}{2}$

$$\| $$

$$\int_0^1 x \, dx$$

\#

$R_f(P) \leftarrow \boxed{C_i}$

Example

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$[x_{i-1}, x_i] \cap \mathbb{Q} \neq \phi$

$\cap \mathbb{Q}^c \neq \phi$

is NOT Riemann integrable (from 0 to 1)

$\exists\, c_i, c_i' \in [x_{i-1}, x_i]$ s.t.

$$\lim R_f(P) = 1$$

$$\lim R_f'(P) = 0 \quad \cancel{\#}$$

<u>Thm</u> (§5.3, §5.4, §5.8) $\alpha, \beta \in \mathbb{R}$

Suppose f and g are integrable on $[a,b]$

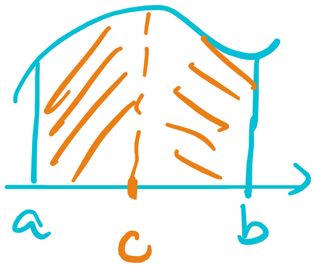
(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^a f(x)\,dx = 0$

(iii) $\int_a^b \alpha f(x) + \beta g(x)\,dx$

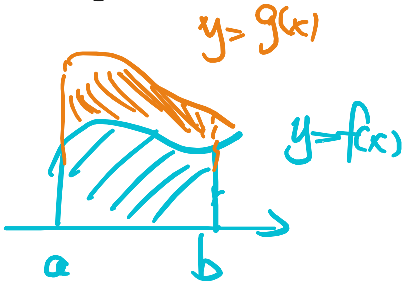
$$= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

(iv) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



(v) If $m \leq f(x) \leq M \quad \forall x \in [a,b]$, then

$$\underset{\displaystyle \overset{\shortparallel}{m(b-a)}}{\int_a^b m\,dx} \leq \int_a^b f(x)\,dx \leq \underset{\displaystyle \overset{\shortparallel}{M(b-a)}}{\int_a^b M\cdot dx}$$



(vi) If $f(x) \geq g(x) \quad \forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

微積分基本定理

<u>Thm</u> (Fundamental Thm of Calculus)

Thm 5.3.5, Thm 5.4.2

Suppose f is continuous on $[a,b]$

(i) The function

$$F(x) := \int_a^x f(t)\,dt, \qquad x \in [a,b].$$

is Continuous on $[a,b]$ and differentiable on (a,b).

Furthermore,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$\forall\ x \in (a,b)$.

(ii) Suppose $G \in C[a,b]$, differentiable on (a,b).

If $G'(x) = f(x)\ \ \forall x \in (a,b)$, then

$$\int_a^b f(x)\,dx = \int_a^b G'(x)\,dx$$

$$= G(b) - G(a)$$

Such a function G is called an antiderivative of f

pf

$$\int_a^x f(t)dt + \int_x^{x+h} f(t)dt$$

$\shortparallel$

(i) Note

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Since f is continuous on $[x, x+h] \subseteq [a,b]$

(by the extreme value thm for continuous functions)

$\exists \; x_{m,h}, x_{M,h} \in [x, x+h]$ s.t.

- $m_h = f(x_{m,h})$ is the min. value of f on $[x, x+h]$
- $M_h = f(x_{M,h})$ is the max. value of f on $[x, x+h]$

$\Rightarrow$

$$\underbrace{\frac{1}{h}\int_x^{x+h} m_h\,dt}_{\frac{1}{h} m_h (x+h-x) \;=\; m_h \;\searrow\; f(x)} \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq \underbrace{\frac{1}{h}\int_x^{x+h} M_h\,dt}_{M_h \;\swarrow\; f(x)} \quad \text{as } h \to 0$$

Furthermore, since f is continuous

$$\lim_{h\to 0} m_h = \lim_{h\to 0} f(x_{m,h}) = f\Big(\lim_{h\to 0} \underbrace{\overset{x+h}{\overset{\vee\!|}{x_{m,h}}}}_{\overset{\wedge\!|}{x}}\Big)$$

$$= f(x)$$

and

$$\lim_{h\to 0} M_h = \lim_{h\to 0} f(x_{M,h}) = f\Big(\lim_{h\to 0} x_{M,h}\Big)$$

$$= f(x)$$

So by the pinching thm,

$$\underbrace{\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt}_{F'(x)} = f(x)$$

That is F is differentiable on (a,b)

That is,

and $F'(x) = f(x)$ $\quad\Downarrow$ Continuous on (a,b)

It remains to show

$$\lim_{x\to a^+} F(x) = F(a) = \int_a^a f(t)\,dt = 0$$

We prove this one

$$\lim_{x\to b^-} F(x) = F(b) = \int_a^b f(t)\,dt$$

Since f is continuous on $[a,b]$, $\exists\, m, M$ s.t.

$$m \le f(x) \le M \qquad \forall x\in[a,b]$$

Thus,

$$\underbrace{m\,(b-x)}_{\to 0} \le \int_x^b f(t)\,dt \le \underbrace{M\cdot(b-x)}_{\to 0} \qquad \forall x\in[a,b].$$

as $x\to b^-$

By pinching thm,

$$\lim_{x\to b^-} \underbrace{\int_x^b f(t)\,dt}_{=\,F(b)-F(x)} = 0$$

$\int_a^{x} f(t)\,dt \qquad \int_a^{b} f(t)\,dt$

$$\Rightarrow \lim_{x\to b^-} F(x) = \lim_{x\to b^-}\Big(\underbrace{F(b)}_{\to F(b)} - \underbrace{(F(b) - F(x))}_{\to 0}\Big)$$

$$= F(b)$$

$\Rightarrow F$ is continuous at b

Similarly, F is continuous at a

$\Rightarrow$ (i) #

(ii) Suppose $G'(x) = f(x) \left(\overset{\text{by (i)}}{=} F'(x)\right)$ $\forall x \in (a,b)$

$\Rightarrow \exists\, C \in \mathbb{R}$ s.t.

$$G(x) = F(x) + C \quad \forall x \in (a,b)$$

$\forall x \in [a,b]$

Furthermore, since F and G are continuous on $[a,b]$,

$$G(a) = \lim_{x\to a^+} G(x) = \lim_{x\to a^+}(F(x)+C) = F(a)+C$$

$$G(b) = \lim_{x \to b^-} G(x) = \lim_{x \to b^-} F(x) + C = F(b) + C$$

$\Rightarrow$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

$$= (F(b) + C) - (F(a) + C)$$

$$= G(b) - G(a)$$

$\Rightarrow$ (ii) #

Example

① $$\frac{d}{dx} \int_0^x \sin(\pi t)\,dt \Big|_{x=\frac{3}{4}}$$

$$= \sin(\pi x) \Big|_{x=\frac{3}{4}} = \sin\left(\frac{3}{4}\pi\right) = \frac{\sqrt{2}}{2}$$ #

② $$\frac{d}{dx} \int_x^0 \sin(\pi t)\,dt = \frac{d}{dx}\left(- \int_0^x \sin(\pi t)\,dt\right)$$

$$= -\sin(\pi x)$$ #

③ $$\frac{d}{dx}\int_0^{x^2} \sin(\pi t)\,dt \overset{\substack{y=x^2 \\ \text{chain rule}}}{=} \frac{d}{dy}\int_0^{y} \sin(\pi t)\,dt \cdot \frac{dy}{dx}$$

$$= \sin(\pi y) \cdot 2x = \sin(\pi x^2) \cdot 2x$$

#

④ $$\int_1^4 x^2\,dx$$

Note: $\underset{\substack{\| \\ G(x)}}{}\left(\frac{x^3}{3}\right)' = x^2$

$$= \left.\frac{x^3}{3}\right|_1^4 = \left.\frac{x^3}{3}\right|_{x=4} - \left.\frac{x^3}{3}\right|_{x=1}$$

$$= \frac{4^3}{3} - \frac{1}{3} = 21$$

#

⑤ $$\int_0^{\frac{\pi}{2}} \sin x\,dx$$

Note:

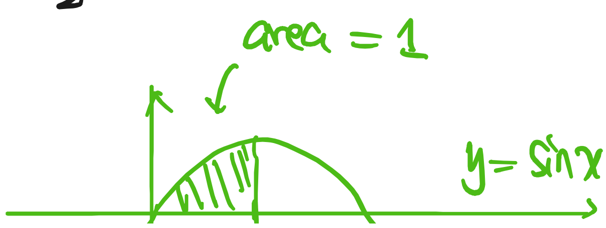
$(\cos x)' = -\sin x$

$(-\cos x)' = \sin x$

$$= \left.-\cos x\right|_0^{\frac{\pi}{2}}$$

$$= -\cos\frac{\pi}{2} - (-\cos 0) = 1$$

#



(6) $$\int_0^1 x^{\frac{3}{2}} - x^{\frac{1}{2}} \, dx$$

$$= \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}} \right) \Big|_0^1$$

$$= \frac{2}{5} - \frac{2}{3} = -\frac{4}{15} \quad \#$$

Note:
$$\left(x^{\frac{p}{q}}\right)' = \frac{p}{q} \cdot x^{\frac{p}{q}-1}$$

$$\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right)' = x^{\frac{3}{2}}$$

$$\left(\frac{2}{3} x^{\frac{3}{2}} \right)' = x^{\frac{1}{2}}$$

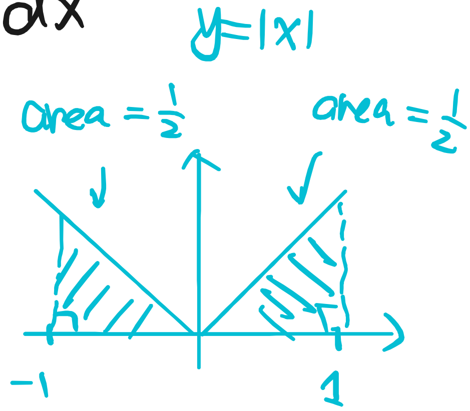
(7) $$\int_{-1}^1 |x| \, dx = \int_{-1}^0 \underset{\substack{\parallel \\ -x}}{|x|} \, dx + \int_0^1 \underset{\substack{\parallel \\ x}}{|x|} \, dx$$

$$= -\int_{-1}^0 x \, dx + \int_0^1 x \, dx$$

$$= -\frac{x^2}{2} \Big|_{-1}^0 + \frac{x^2}{2} \Big|_0^1$$

$$= -\Big(\underbrace{\frac{0^2}{2} - \frac{(-1)^2}{2}}_{-\frac{1}{2}} \Big) + \left(\frac{1^2}{2} - \frac{0^2}{2} \right)$$

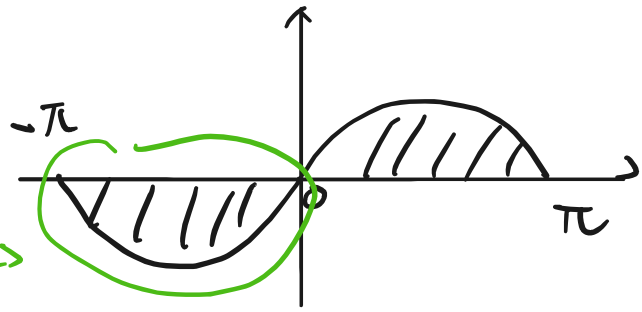
$$= \frac{1}{\ } + \frac{1}{\ } = 1$$



$- \quad 2 \quad ' \quad 2 \quad ' \quad \#$

⑤ Find the area of the region:

$y = \sin x$



$-\int_{-\pi}^{0} \sin x \, dx$

Note: $\int_{-\pi}^{\pi} \sin x \, dx = 0$

Sol

$$\text{area} = -\int_{-\pi}^{0} \sin x \, dx + \int_{0}^{\pi} \sin x \, dx$$

$$= +\, (+\cos x)\Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi}$$

$$= \big(1 - (-1)\big) + \big(-(-1) - (-1)\big)$$

$$= 4 \quad \#$$

Remark (Indefinite integral, §5.6)

Some people write

$$\int f(x)\,dx = F(x) + C$$

if $F'(x) = f(x)$.

e.g.

$$\int x\,dx = \frac{1}{2}x^2 + C$$

<u>Thm</u> (Integration by substitution / u-substitution, Thm 5.7.1, Eg. 5.7.2)

If f is continuous, $F' = f$ and u is differentiable, then

$$\int_a^b f(u(x)) \cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

<u>pf</u>

By chain rule,

$$\frac{d}{dx} F(u(x)) = F'(u(x)) \cdot u'(x)$$

$$= f(u(x)) \cdot u'(x)$$

$$\Rightarrow \int_a^b f(u(x)) \cdot u'(x)\, dx = F(u(x)) \Big|_{x=a}^{b}$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} f(u)\, du$$

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Convention notation;

$$\text{" } \underline{\underline{du}} = \underline{\underline{u'(x)}}\, dx = \frac{du}{dx} \cdot dx \text{ "}$$

$$\int f(u(x))\, \underline{\underline{u'(x)\, dx}} = \int f(u)\, \underline{\underline{du}}$$

Example

① $\int_0^1 \frac{1}{(3+5x)^2}\, dx = ?$

Let $u = 3+5x \Rightarrow du = 5dx$

$$\int^{u(1)=8} \quad \frac{1}{} \qquad \int^{8} \text{"} \frac{1}{5} u^{-2}$$

$$= \int_{u(0)=3} \frac{1}{u^2}\,\frac{1}{5}\,du = \int_3 \overline{5u^2}\,du$$

$$= \frac{1}{5}\,\frac{u^{-1}}{-1}\Bigg|_{u=3}^{8} = -\frac{1}{5}\left(\frac{1}{8}-\frac{1}{3}\right)$$

$$= \frac{1}{24} \quad \#$$

② $\displaystyle\int_0^1 x^2\sqrt{4+x^3}\,dx = ?$ $\qquad = \frac{1}{3}du$

Let $u = 4+x^3 \Rightarrow du = 3x^2\,dx$

$$= \int_{u(0)=4}^{u(1)=5} \sqrt{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{u=4}^{5}$$

$$= \frac{2}{9}\left(5\sqrt{5}-8\right) \quad \#$$

$= -\frac{1}{\pi}du$

③ $\displaystyle\int_0^{\frac{1}{2}} \cos^3(\pi x)\cdot\sin(\pi x)\,dx = ?$

Let $u = \cos\pi x \Rightarrow du = -\pi\sin\pi x\,dx$

$$= \int_{u(0)=1}^{u(\frac{1}{2})=0} u^3 \left(-\frac{1}{\pi}\right) du = -\frac{1}{\pi} \left. \frac{u^4}{4} \right|_{u=1}^{0}$$

$$= -\frac{1}{\pi}\left(0 - \frac{1}{4}\right) = \frac{1}{4\pi} \quad \#$$

<u>Thm</u> (Integration by parts, §8.2)

Suppose $u, v \in C^1[a,b]$. Then

"$\int u\,\underbrace{dv}_{v'(x)dx} = uv - \int v\,\underbrace{du}_{u'(x)\,dx}$"

$$\int_a^b u(x)\cdot v'(x)\,dx$$

$$= \left. u(x)\cdot v(x) \right|_{x=a}^{b} - \int_a^b v(x)\cdot u'(x)\,dx$$

pf

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

$$\Rightarrow \underbrace{\int_a^b (u(x)\cdot v(x))'\,dx}_{\shortparallel} = \int_a^b u'(x)\cdot v(x)\,dx + \int_a^b u(x)\cdot v'(x)\,dx$$

$$u(x)\cdot v(x)\Big|_{x=a}^{b}$$

$$\Rightarrow \int_a^b u(x)\cdot v'(x)\,dx = u(x)\cdot v(x)\Big|_{x=a}^{b} - \int_a^b v(x)\cdot u'(x)\,dx$$

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Example $u(x)\cdot v'(x)$

Let
$u(x) = x \qquad u'(x) = 1$

$v(x) = -\cos x \qquad v'(x) = \sin x$

① $\displaystyle\int_0^{\pi} x\cdot\sin x\,dx$

$$= \underbrace{x(-\cos x)\Big|_0^{\pi}}_{\pi(+1) - 0(-\cos 0) = \pi} - \int_0^{\pi} 1\cdot\underbrace{(-\cos x)}_{(-\sin x)'}\,dx$$

$$= \pi - \cancel{(-\sin x)\Big|_0^{\pi}} \Rightarrow 0 \quad = \pi$$

#

Let
$u(x) = x \qquad u'(x) = 1$

$v(x) = \frac{2}{3}(x+1)^{\frac{3}{2}} \qquad v'(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

② $\displaystyle\int_0^1 \overset{u(x)}{x}\,\overset{v'(x)}{\sqrt{x+1}}\,dx$

$$= x\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|^1 - \int^1 \frac{2}{3}(x+1)^{\frac{3}{2}} \ldots$$

$$\Big|_0 \qquad \int_0 \Big(\frac{\;}{3}(x+1)\Big)\, dx$$

$$\frac{2}{3}\, 2^{\frac{3}{2}} - 0 = \frac{4\sqrt{2}}{3} \qquad\qquad \left(\frac{2}{3}\cdot\frac{2}{5}(x+1)^{\frac{5}{2}}\right)'$$

$$= \frac{4\sqrt{2}}{3} - \frac{4}{15}(x+1)^{\frac{5}{2}}\Big|_0^1$$

$$= \frac{4\sqrt{2}}{3} - \frac{4}{15}\Big(2^{\frac{5}{2}} - 1\Big) \qquad 2^{\frac{5}{2}} = 4\sqrt{2}$$

$$= \frac{4}{3}\sqrt{2} - \frac{16}{15}\sqrt{2} + \frac{4}{15} \quad \#$$

More properties of integrals

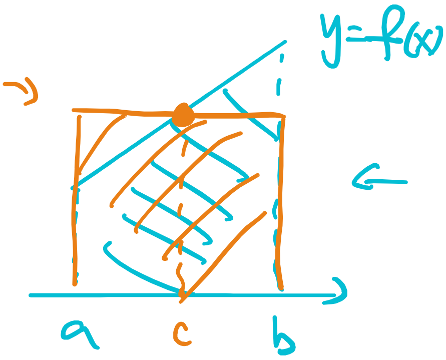
Thm (First mean value thm for integrals, Thm 5.9.1)

If $f \in C[a,b]$, then $\exists\, c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

called average value or mean value

$f(c)\cdot(b-a)$ →



← $\int_a^b f(x)\,dx$ of f on $[a,b]$

pf

Let $F(x) = \int_a^x f(t)\,dt.$

By MVT, $\exists\ c \in (a,b)$ s.t.

$$F'(c) = \frac{\overbrace{F(b)}^{\int_a^b f(t)dt} - \overbrace{F(a)}^{=0}}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\|$

$f(c)$

#

Thm (§5.8)

Suppose $\underbrace{f, g}_{|f|,\ |g|}$ are integrable.

(i) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

y=sinx

area $= \int^{\pi} |\sin x|\,dx$

$$\int_{-\pi}^{\pi} \sin x \, dx = 0$$

(ii) If $f \in C[a,b]$, u is differentiable, then

$$\frac{d}{dx} \int_a^{u(x)} f(t)\, dt = f(u(x)) \cdot u'(x)$$