Calculus, Fall 2025, week 10

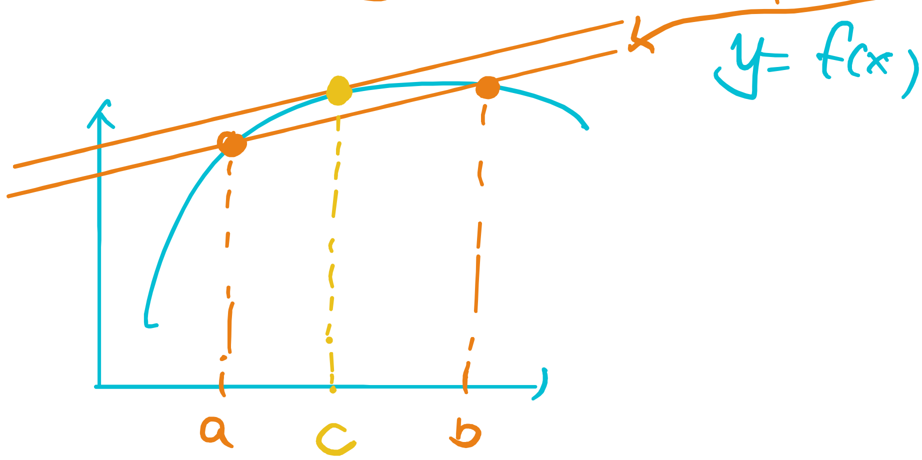
Recall (Mean Value Thm)

If f is differentiable on (a,b) and continuous on $[a,b]$.

then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

slope of



Example

Show that $p(x) = 2x^3+5x-1$ has exactly one solution

pf

Step 1 ($\exists\, c$ s.t. $p(c)=0$):

$p(0) = -1 < 0$

$p(1) = 2+5-1 = 6 > 0$

p is continuous

$\Rightarrow$ By the intermediate thm, $\exists\, c \in (0,1)$ s.t.

$$p(c) = 0$$

<u>Step2</u> (No other c' s.t. $p(c') = 0$)

Suppose $c' \neq c$, $p(c') = 0$.

Then by MVT, $\exists\, d$ s.t.

$$p'(d) = \frac{p(c) - p(c')}{c - c'} = 0$$

$$\Vert$$

$$6d^2 + 5 > 0 \qquad (\longrightarrow\longleftarrow)$$

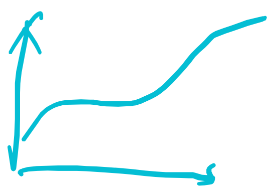
So c is the unique real solution of

$$p(x) = 2x^3 + 5x - 1 = 0 \qquad \boxtimes$$

Monotone functions

<u>Def</u> (Def 4.2.1)

A function f is said to be



(i) increasing on (a,b) if

$$x_1 < x_2,\ x_1, x_2 \in (a,b) \Rightarrow f(x_1) \leq f(x_2)$$

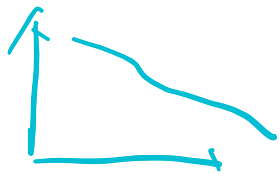
(ii) strictly increasing on (a,b) if

(ii) strictly increasing on (a,b) if

$x_1 < x_2$, $x_1, x_2 \in (a,b) \Rightarrow f(x_1) < f(x_2)$

(iii) decreasing on (a,b) if

$x_1 < x_2$, $x_1, x_2 \in (a,b) \Rightarrow f(x_1) \geq f(x_2)$



(iv) strictly decreasing (a,b) if

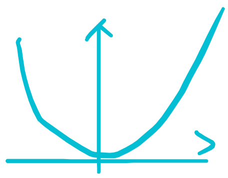
$x_1 < x_2$, x_1, x_2 $(a,b) \Rightarrow f(x_1) > f(x_2)$

(v) Constant on (a,b) if

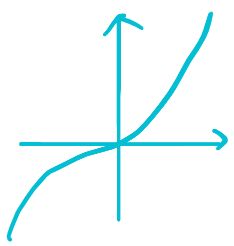
$f(x_1) = f(x_2)$ $\forall x_1, x_2 \in (a,b)$

Example

① $f(x) = x^2$ is decreasing on $(-\infty, 0]$
increasing on $[0, \infty)$

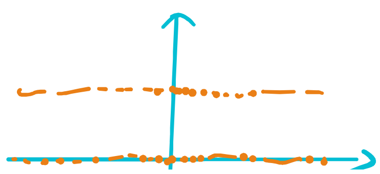


② $f(x) = x^3$ is strictly increasing on $(-\infty, \infty)$



有理數
{rational numbers}
‖

③ $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \end{cases}$



((, $x \notin \mathbb{Q}$

is NOT increasing, nor decreasing on any (a,b)

Thm (Thm 4.2.2, Thm 4.2.3)

Suppose f is differentiable on (a,b)

(i) $f'(x) \geq 0 \quad \forall x \in (a,b) \Rightarrow f$ is increasing on (a,b)

(ii) $f'(x) > 0$ " $\Rightarrow$ " strictly increasing "

(iii) $f'(x) \leq 0$ " $\Rightarrow$ " decreasing "

(iv) $f'(x) < 0$ " $\Rightarrow$ " strictly decreasing "

(v) $f'(x) = 0$ " $\Rightarrow$ " constant "

pf. of (ii) (others are similar)

$\forall x_1, x_2 \in (a,b)$, $x_1 < x_2$, by MVT, $\exists\, c \in (x_1, x_2)$

$\Downarrow$

$c \in (a,b)$

s.t.

$$\underbrace{f'(c)}_{>0} = \frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}} > 0$$

$$\Rightarrow \quad f(x_2) - f(x_1) > 0 \cdot (x_2 - x_1) = 0$$

$$\Rightarrow \quad f(x_2) > f(x_1) \qquad \#$$

Example

① $f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

$\Rightarrow$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

② $g(x) = x - 2\sin x$, $x \in [0, 2\pi]$.

$\Rightarrow g'(x) = 1 - 2\cos x$ $\quad$ ($g'(x) = 0 \Leftrightarrow \cos x = \frac{1}{2}$, $x = \frac{\pi}{3}, \frac{5}{3}\pi$)

$\Rightarrow \cos x < \frac{1}{2}$

x	$(0, \frac{\pi}{3})$	$\frac{\pi}{3}$	$(\frac{\pi}{3}, \frac{5}{3}\pi)$	$\frac{5}{3}\pi$	$(\frac{5}{3}\pi, 2\pi)$
$g'(x)$	$-$		$+$		$-$
$g(x)$	$\searrow$		$\nearrow$		$\searrow$

<u>Thm</u> (Thm 4.2.4)

Suppose f, g are differentiable on (a,b)

(i) $f'(x) = 0 \ \forall x \in (a,b) \Leftrightarrow f(x) = \text{constant}$

(ii) $f'(x) = g'(x) \ \forall x \in (a,b) \Leftrightarrow f(x) = g(x) + C$ for some $C \in \mathbb{R}$.

$\Updownarrow \qquad \Updownarrow$

$(f-g)'(x) = 0 \Leftrightarrow f - g = \text{constant}$

Extreme values

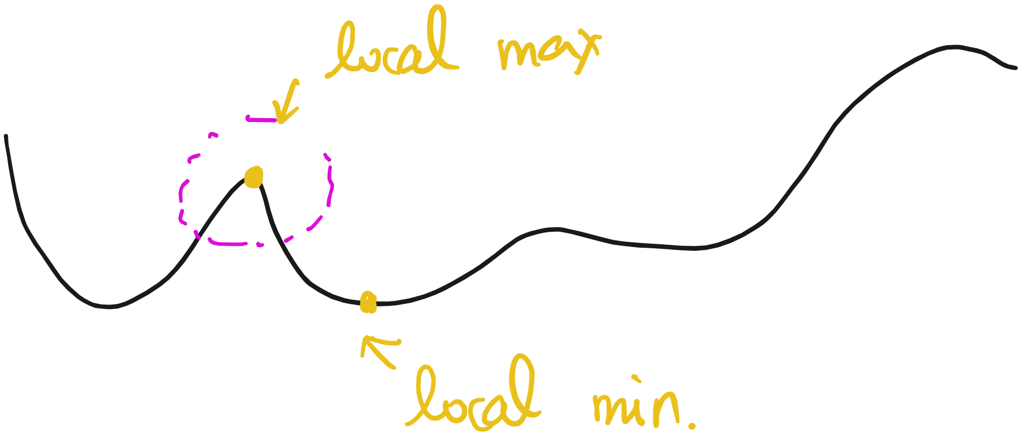
local minimum

Def (Def 4.3.1)

A function f has a local maximum value at c if $\exists\ \varepsilon > 0$ s.t.

$$f(c) \geq f(x) \quad \forall\, x \in (c-\varepsilon,\ c+\varepsilon)$$

$$f(c) \leq f(x) \quad \forall\, x \in (c-\varepsilon,\ c+\varepsilon)$$

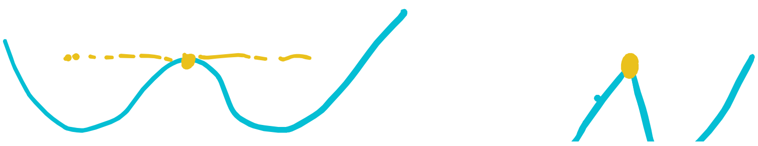


Remark

If f has an (absolute) maximum (minimum) value at c, then f has a local maximum (minimum) value at c.

Thm (Thm 4.3.2)

If f has a local maximum / minimum at c then either $f'(c) = 0$ or $f'(c)$ doesn't exist.



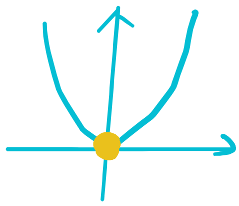
Def (Def 4.3.3)

A point c is called a critical point for f if $f'(c) = 0$ or $f'(c)$ doesn't exist.

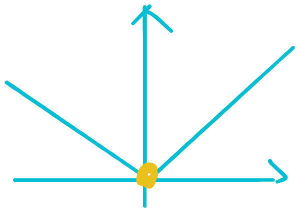
Example

① $f(x) = x^2 \Rightarrow f'(x) = 2x$

critical point of f: $x = 0$



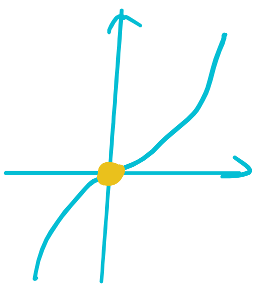
② $f(x) = |x| \Rightarrow f'(x) \begin{cases} 1 & x > 0 \\ \text{doesn't exist} & x = 0 \\ -1 & x < 0 \end{cases}$



critical point of f: $x = 0$

③ $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

critical point of f: $x = 0$



Q: If c is a critical point of f, how do we know f has a local max/min at c !

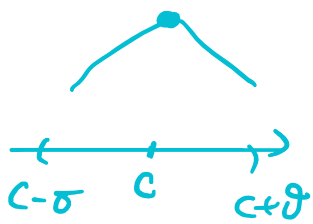
Thm (Thm 4.3.4)

Suppose c is a critical point of f.
f is continuous at c.

If $\exists \delta > 0$ s.t.

(i) $f'(x) > 0 \ \forall x \in (c-\delta, c)$

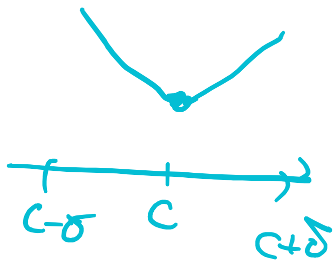
$f'(x) < 0 \ \forall x \in (c, c+\delta)$



$\Rightarrow$ f has a local maximum at c

(ii) $f'(x) < 0 \ \forall x \in (c-\delta, c)$

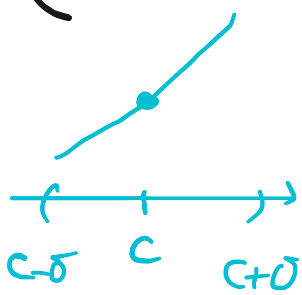
$f'(x) > 0 \ \forall x \in (c, c+\delta)$



$\Rightarrow$ f has a local minimum at c

(iii) $f'(x) > 0 \quad \forall x \in (c-\delta, c+\delta)$

or $f'(x) < 0 \quad \forall x \in (c-\delta, c+\delta)$



$\Rightarrow$ f does NOT have a local extreme at c.

Thm (Thm 4.3.5)

Suppose $f''(c)$ exist, f' exists near c, $f'(c) = 0$

(i) $f''(c) > 0 \Rightarrow$ f has a local minimum at c

(ii) $f''(c) < 0 \Rightarrow$ f has a local maximum at c

(iii) $f''(c) = 0 \Rightarrow$ anything can happan.

Example

① $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 > 0$

$\Rightarrow$ f has a local minimum at $x = 0$

② $f(x) = |x| \Rightarrow f'(0)$ does NOT exist

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	does NOT exist	$+$
$f(x)$	↘		↗

$$f'(x) \begin{cases} 1 & x > 0 \\ \text{doesn't exist} & x = 0 \\ -1 & x < 0 \end{cases}$$

$\Rightarrow$ f has a local minimum at $x = 0$.

③ $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ $\quad f''(x) = 6x$

$f'(0) = 0$ $\quad f''(0) = 0$

x | $(-\infty, 0)$ $\quad 0 \quad (0, \infty)$

x				
$f'(x)$	$+$	0	$+$	
$f(x)$	$\nearrow$	$\bullet$	$\nearrow$	

f does NOT have a local extreme at $x=0$

Example

$$f(x) = x^4 - 2x^3$$

Q: Find local max/min.

Sol

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$$

$f'(x)=0 \iff x=0$ or $\frac{3}{2}$ ← critical points

x	$(-\infty, 0)$	0	$(0, \frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2}, +\infty)$
$f'(x)$	$-$		$-$		$+$
$f(x)$	$\searrow$		$\searrow$	$\cdot$	$\nearrow$

$\Rightarrow$ $f(\frac{3}{2})$ is a local mini

$\Rightarrow$ $f(\frac{1}{2})$ is a local minimum

<u>Recall</u>:

- local max/min. : $\exists\, \delta > 0$

$$f(c) \underset{(\leq)}{\geq} f(x) \qquad \forall\, x \in (c-\delta, c+\delta)$$

- (global/absolute) max/min. :

定義域

$$f(c) \underset{(\leq)}{\geq} f(x) \qquad \forall x \in \text{domain}(f)$$

<u>Example</u>

Find the critical points of the following functions and classify all the extreme values.

① $f(x) = 1 + 4x^2 - \frac{1}{2}x^4, \qquad x \in [-1, 3]$

<u>Sol</u> (Need to endpoints)

<u>Step1</u> (Critical points) $\qquad 2x(2-x)(2+x)$

$$f'(x) = 8x - 2x^3 = \underline{2x(4-x^2)} = 0$$

$$\Leftrightarrow \quad x = 0, \pm 2 \qquad \text{Note: } -2 \notin [-1, 3]$$

So $x = 0, 2$ are critical points

Step 2: (Find local max/min in the interior $(-1, 3)$)

x	-1		0		2		3
$f'(x)$		$-$		$+$		$-$	
$f(x)$	local max	$\searrow$	local min	$\nearrow$	local max	$\searrow$	local min

Step 3: Add endpoints. Compare all the values

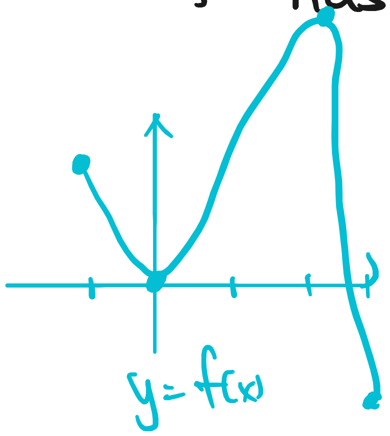
$$f(-1) = \frac{9}{2}$$

$$f(0) = 0$$

$$f(2) = 9 \quad \leftarrow \text{global max.}$$

$$f(3) = -\frac{7}{2} \quad \leftarrow \text{global min.}$$

Conclusion:



f has local min. at 0 and 3

local max at −1 and 2

global min. at 3

global max at 2

#

② $f(x) = \frac{1}{4}\left(x^3 - \frac{3}{2}x^2 - 6x + 2\right), \quad x \in [-2, \infty)$

sol

$$f'(x) = \frac{3}{4}x^2 - \frac{3}{4}x - \frac{6}{4} = \frac{3}{4}(x^2 - x - 2)$$

$$= \frac{3}{4}(x-2)(x+1)$$

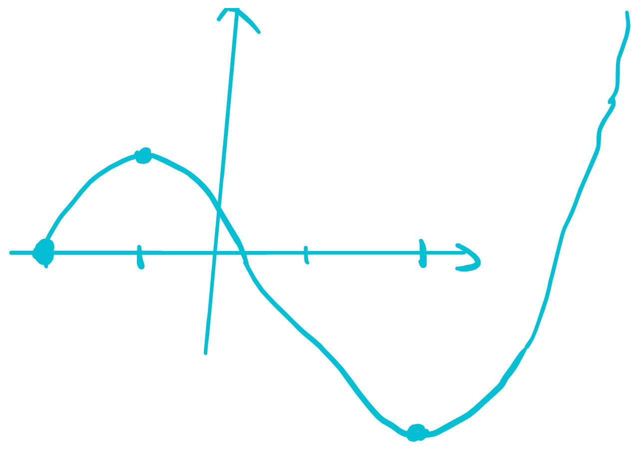
$$f'(x) = 0 \iff x = -1, 2$$

x	-2		-1		2	
$f'(x)$		$+$		$-$		$+$
$f(x)$	local min	$\nearrow$	local max	$\searrow$	local min	$\nearrow$

$f(-2) = 0$

$f(-1) = \frac{11}{8}$

$f(2) = -2 \leftarrow$ global min.



$f(10) > f(-1) = \frac{11}{8} \leftarrow$ NOT global max.

Conclusion:

f has local min at -2, 2

local max at -1

global min at 2

NO global max.

#

<u>Def</u> (Def 4.6.1, 4.6.2)

The graph of a differentiable function f is

(i) <u>concave up</u> if $f'(x)$ is increasing

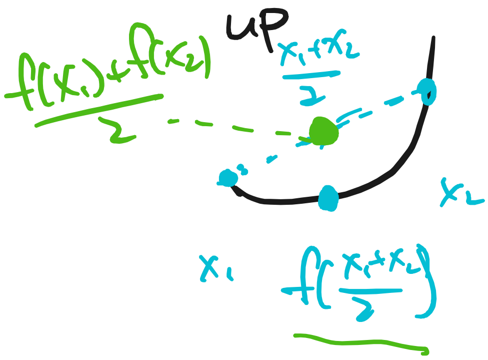
凹向上

(ii) concave down 凹向下 if $f'(x)$ is decreasing

A point $(c, f(c))$ is a point of inflection 反曲點 if $\exists\, \delta > 0$ s.t.

- f is concave up/down on $(c-\delta, c)$
- " " down/up " $(c, c+\delta)$

Concave up



concave down

Thm (Thm 4.6.3, Thm 4.6.4)

If $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or $f''(c)$ does NOT exist.

Suppose f is twice differentiable. Then

(i) $f''(x) > 0 \ \forall x \in (a,b) \Rightarrow$

the graph of f is concave up on (a,b)

(ii) $f''(x) < 0 \ \forall x \in (a,b) \Rightarrow$

the graph of f is concave down on (a,b)

<u>Example</u>

$$f(x) = x^3 - 6x^2 + 9x + 1$$

$$\Rightarrow f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$

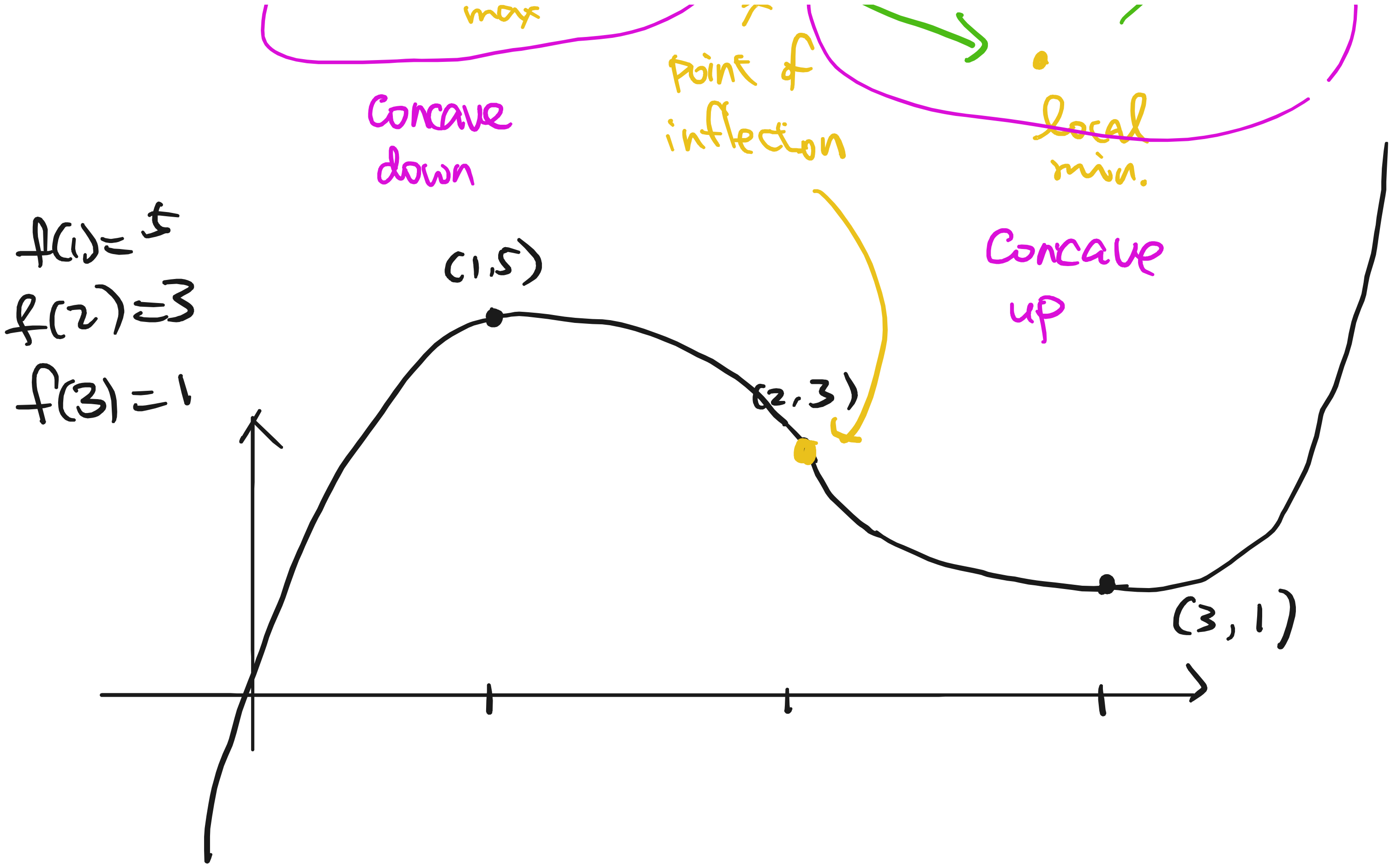
$$= 3\underline{(x-1)(x-3)}$$

$f'(x) = 0 \Leftrightarrow x = 1, 3$

$$f''(x) = 6x - 12 = 6(x-2)$$

$f''(x) = 0 \Leftrightarrow x = 2$

x		1		2		3	
$f''(x)$		$-$		0		$+$	
$f'(x)$	$+$ ↘	0	↘ $-$		$-$ ↗	0	↗ $+$
$f(x)$	↗	• local	↘	•			↗



L'Hôpital's rule

Thm (0/0 type, §11.5)

Suppose $\exists\, \delta > 0$ s.t. f, g are differentiable

$\Downarrow$ continuous $\Downarrow$

on $(c-\delta,\ c+\delta)$ and

$f(c)$ $\qquad\qquad$ $g(c)$

$$\lim_{x \to c} f(x) = \overset{f(c)\,=}{0}, \qquad \lim_{x \to c} g(x) = \overset{g(c)\,=}{0}.$$

If $g'(x) \neq 0 \quad \forall\, x \in (c-\delta, c+\delta),\ x \neq c,$

and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L,$$

e.g. $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} \frac{(x^2)'}{(x)'} \frac{2x}{1}$

$= 0$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

pf

Generalized MVT

Recall (Cauchy's Mean Value Thm)

Suppose f, g are continuous on $[a,b]$

differentiable on (a,b), $g'(x)\neq 0$ $\forall x\in(a,b)$

Then $\exists\, c\in(a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Given $x\in(c-\delta, c+\delta)$, $x\neq c$, since

f, g are differentiable on $(c-\delta, c+\delta)$

they are continuous on $\underbrace{[c,x]}_{\text{if } x>c}$ or $\underbrace{[x,c]}_{\text{if } x<c}$

$\cap$
$(c-\delta, c+\delta)$

and differentiable on (c, x) or (x, c)

So by Cauchy's MVT, $\exists\ \xi \in (c, x)$ or (x, c)
$\xi = \xi(x)$

s.t.

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)}$$

with $f(c) = 0$ and $g(c) = 0$, so the left side $= \frac{f(x)}{g(x)}$.

As $x \to c$, the corresponding $\xi = \xi(x)$ lies between x and c, so $\xi(x) \to c$.

$$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{t\to c} \frac{f'(t)}{g'(t)}$$

$$= L \quad \#$$

Example L'Hôpital's rule

Example

L'Hôpital's rule

① $$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi-2x} \quad \left(\frac{\to 0}{\to 0}\right) = \lim_{x\to\frac{\pi}{2}} \frac{(\cos x)'}{(\pi-2x)'}$$

$(\cos x)' = -\sin x$

$(\pi-2x)' = -2$

$$= \frac{-\sin\frac{\pi}{2}}{-2} = \frac{1}{2} \quad \#$$

L'Hôpital

② $$\lim_{x\to 1} \frac{x^4+2x^3-2x^2-1}{x^3-1} \quad \left(\frac{\to 0}{\to 0}\right) = \lim_{x\to 1} \frac{4x^3+6x^2-4x}{3x^2}$$

$4+6-4=6$

$3x^2 \to 3 \neq 0$

$$= \frac{6}{3} = 2 \quad \#$$

$f(1) \le 1$

Show that if $f \in C[0,1]$, $f([0,1]) \subseteq [0,1]$, then $\exists\, c \in [0,1]$ s.t. $f(c) = c$.

Q1.

不清楚中間值定理的定理內容，以此題為例：

$\ge$

$h(0) = f(0) - 0 \ge 0$

- $h(x) = f(x) - x$ 函數要在 $[0,1]$ **連續**

  （特別提醒，$h(x)$不一定會在實數上都連續）
- $h(1) < 0 < h(0)$，不能等於 0 $\le$
- 存在的零點會落在 (0,1) 裡，而非 [0,1]

case1: $f(0) = 0$ or $f(1) = 1$ $\Rightarrow$ ok ($c = 0$ or 1)

If f is continuous on $[a,b]$ and $f(a) < k < f(b)$ or $f(a) > k > f(b)$ then $\exists\, c \in (a,b)$ s.t. $f(c) = k$.

Q3. Prove that $f(x) = x + x^2 + \cdots + x^n = \dfrac{x - x^{n+1}}{1-x}$

有少數人會從 $f(x) = \frac{x-x^{n+1}}{1-x}$ 開始討論，再把兩邊同乘 $1-x$，但這樣會變成你先假設結論是對再去討論。

case2: $f(0) \neq 0$ and $f(1) \neq 1$

Consider $h(x) = f(x) - x$

$\Rightarrow h(0) = f(0) - 0 > 0 \Rightarrow \cdots$

$h(1) = f(1) - 1 < 0$

$f(x) = x + \cdots + x^n$

$\Rightarrow x f(x) = \cancel{x^2} + \cdots + \cancel{x^n} + x^{n+1}$

$\Rightarrow f(x) - x f(x) = x - x^{n+1}$

$\Rightarrow f(x) = \dfrac{x - x^{n+1}}{1-x} \quad \forall x \neq 1$

Average	40.4347826			
Average (except zero)	44.2857142			
Quartile 0	0			
Quartile 1	28.5			
Quartile 2	48			
Quartile 3	55			
Quartile 4	60			
標準差	19.0894158			
非零數	105			
不到50%人數	30	0	0	0
滿分人數	10	0	0	0