Calculus, Fall 2025, week 1

Introduction

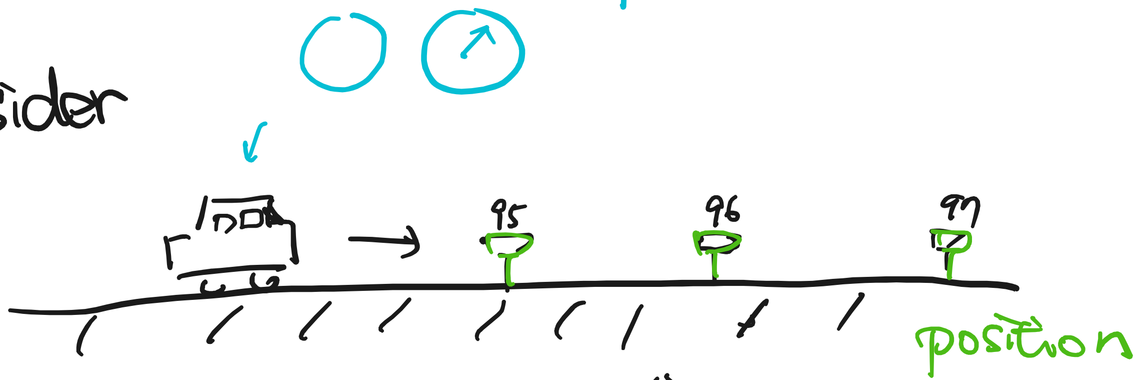
Main topics:

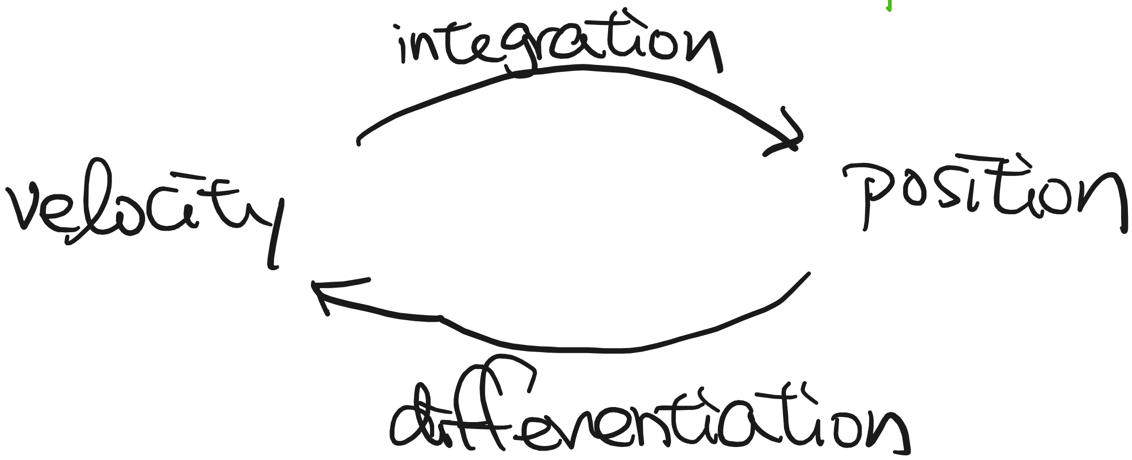
"differentiation" and "integration"

微分 積分

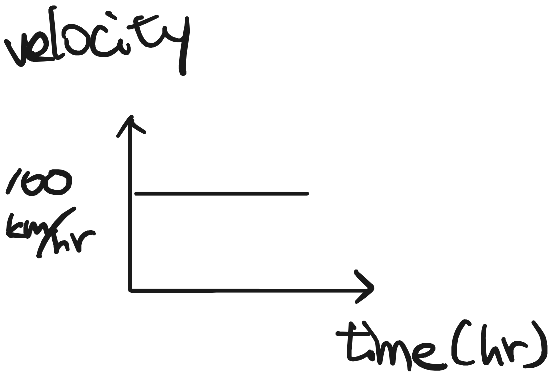
Consider

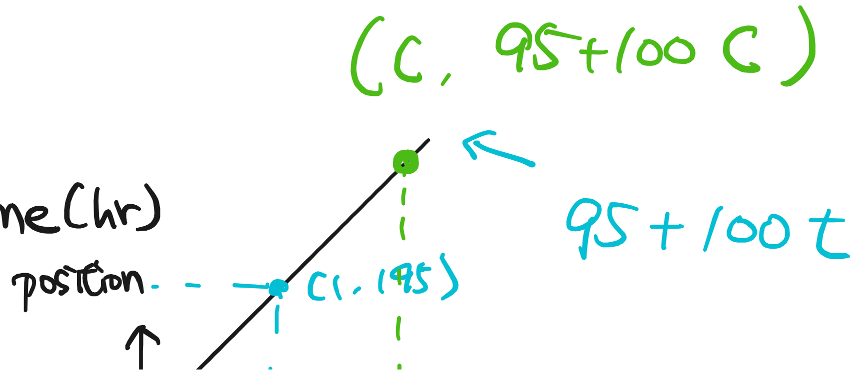
velocity



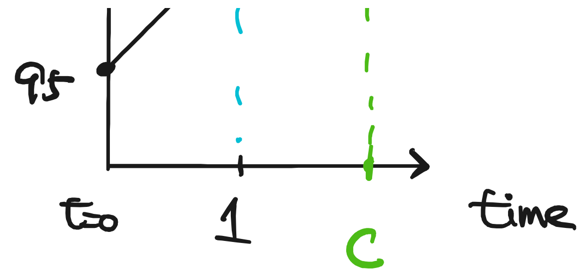


case1:





$\Rightarrow$

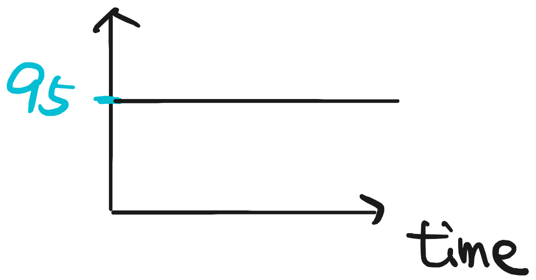


$$\Rightarrow \quad 95 + \int_0^c 100 \, dt = 95 + 100c$$

over simplified

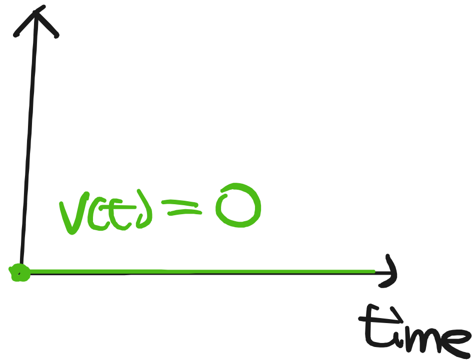
case 2:

position



$\Rightarrow$

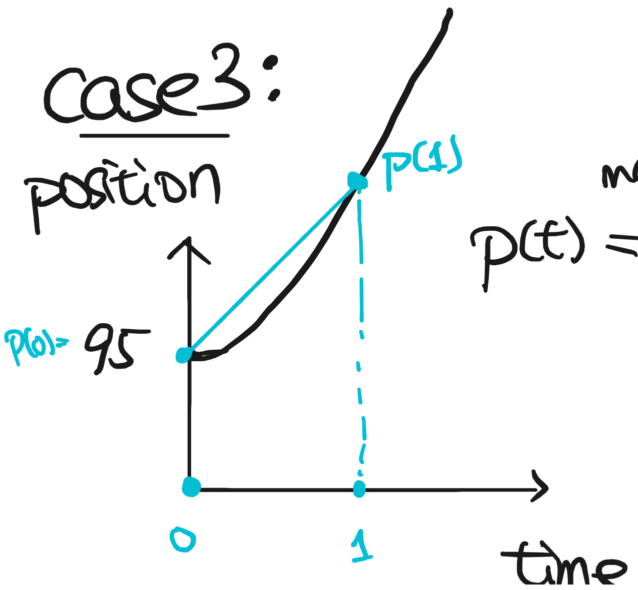
velocity



$$\Rightarrow \quad \frac{d}{dt}(95) = 0$$

case 3:

position



$$p(t) = \overset{\text{maybe}}{95 + 100t \cdot 2^{-\frac{1}{t}}} \quad ?$$

Q: $V(t)=$ velocity at $t=$?

for example

difficult, but we have an approximation.

e.g.

$V(1)=$?

an approximation:

$$\frac{P(1)-P(0)}{1-0}$$ ← 可能和 $V(1)$ 差不多

$$\frac{P(1)-P(0.9)}{1-0.9}$$ ← 可能和 $V(1)$ 更接近

$$V(1)=\lim_{t\to 1}\frac{P(1)-P(t)}{1-t}$$

$$=\left.\frac{d}{dt}P(t)\right|_{t=1}$$

Limits of sequences

Limits of sequences

We will see the following 3 types of limits:

- function-type A:

$$\lim_{x \to c} f(x) \qquad (c \text{ is a fixed number})$$

- function-type B:

$$\lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x)$$

- sequence type: ← more fundamental

$$\lim_{n \to \infty} \underline{a_n}$$

← sequence 數列

e.g.

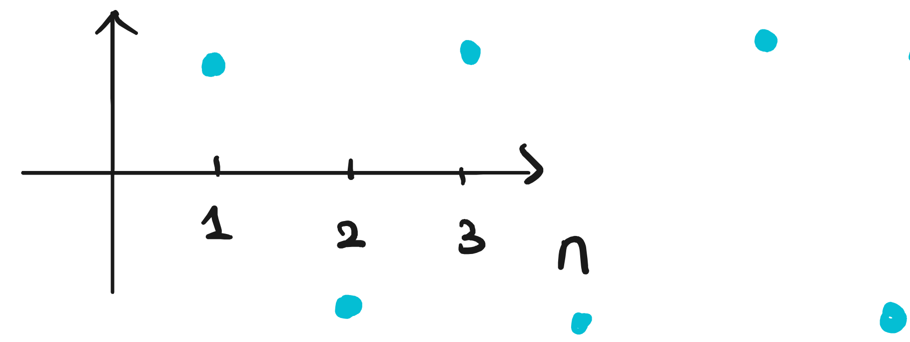
① $\underset{a_1}{1}, -1, 1, -1, \ldots$

② $b_1 = \frac{1}{1},\ b_2 = \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots$

Example

$(1,1) = (1, a_1)$ $\quad (3,1) = (3, a_3)$

①

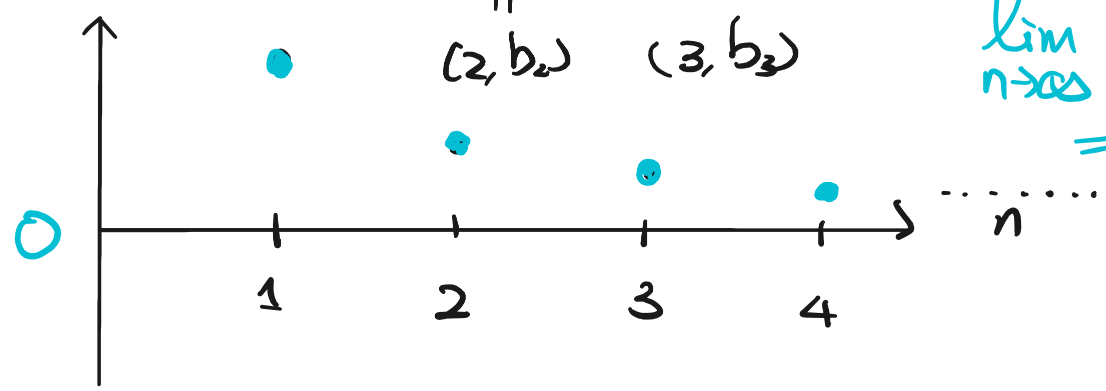


$(2,-1) = (2, a_2)$

$\lim_{n\to\infty} a_n$ doesn't exist

$(1,1) = (1, b_1)$ $\quad (2, \frac{1}{2}) = (2, b_2)$ $\quad (3, \frac{1}{3}) = (3, b_3)$

②



guess: $\lim_{n\to\infty} b_n = 0$

Fundamental questions:

(i) Does $\lim_{n\to\infty} a_n$ exist?

(ii) If $\lim\limits_{n\to\infty} a_n$ exists,

$$\lim_{n\to\infty} a_n = ?$$

Need to be sure!!

To discuss these questions, we need to know the precise meaning of $\lim\limits_{n\to\infty} a_n$.

Definition of sequence

Recall that a function (函數) $f: A \to B$ has 3 ingredients:

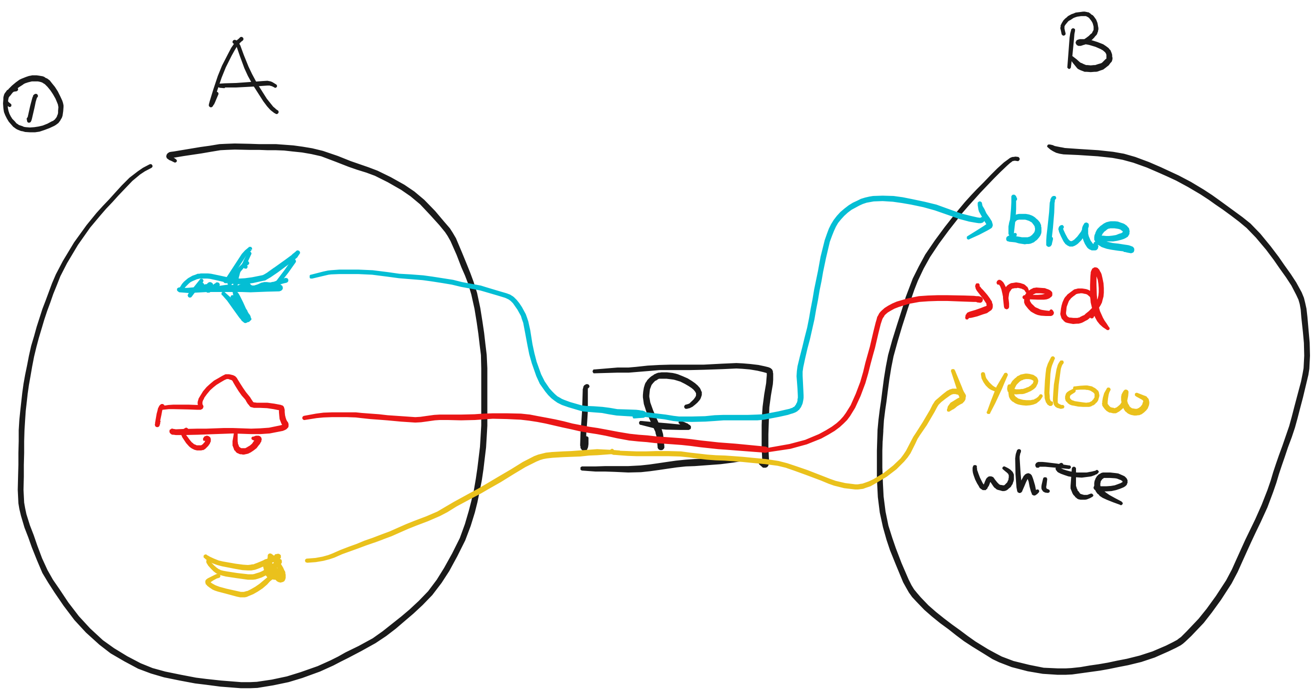
- domain (定義域) $= A =$ the set of possible inputs

對應域

- codomain $= B =$ the set of possible outputs

- how it works

eg. ①



② $f(x) = \dfrac{1}{x^2-1}$ ← how it works

屬於　滿足

$f: \mathbb{R} - \{\pm 1\} \longrightarrow \mathbb{R}$

$= \{x \in \mathbb{R} \mid x \notin \{\pm 1\}\}$

不屬於

$= \{$除±1外的所有實數$\}$

$\mathbb{R} = \{$real numbers$\}$

集合

= the set of real numbers
實 數

Definition = 定義

<u>Def</u> (Def 11.2.1) $a : \mathbb{N} \to \mathbb{R}$

A sequence (of real numbers) is a function from $\mathbb{N}$ to $\mathbb{R}$,

where

正 整 數

$\mathbb{N}$ = the set of positive integers

$$= \{1, 2, 3, 4, 5, \cdots, n, n+1, \cdots\}$$

If $a : \mathbb{N} \to \mathbb{R}$ is a sequence,

we usually write

$$a(n) = a_n ,$$

第 n 項

called the n-th term of this

sequence

sequence.

It is common to denote $a : \mathbb{N} \to \mathbb{R}$ by

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

Example

① $$(a_n)_{n=1}^{\infty} = \underset{a_1}{1}, \underset{a_2}{-1}, \underset{a_3}{1}, -1, 1, -1, \dots$$

$$= \Big(\underbrace{(-1)^{n+1}}_{a_n}\Big)_{n=1}^{\infty}$$

② $$(b_n)_{n=1}^{\infty} = \underset{b_1}{\frac{1}{1}}, \underset{b_2}{\frac{1}{2}}, \frac{1}{3}, \dots$$

$$= \left(\frac{1}{n}\right)_{n=1}^{\infty}, \quad \frac{1}{n} = b_n$$

③ $$(c_n)_{n=1}^{\infty} = 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

$$\begin{cases} C_1 = C_2 = 1, \\ C_n = C_{n-1} + C_{n-2} & \forall\, n \geq 3 \end{cases}$$

for all

$\lim_{n\to\infty} a_n$ exists

$\Updownarrow$

4條敘句

<u>Def</u>

A sequence $(a_n)_{n=1}^{\infty}$ is convergent if there exists a number $L \,(\in \mathbb{R})$ with the following property:

"誤差"

for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$|a_n - L| < \varepsilon$$

$a_N,\ a_{N+1},\ a_{N+2}, \dots$

whenever $n \geq N = N(\varepsilon)$

In this case, we say L is the limit of $(a_n)_{n=1}^{\infty}$, denoted by

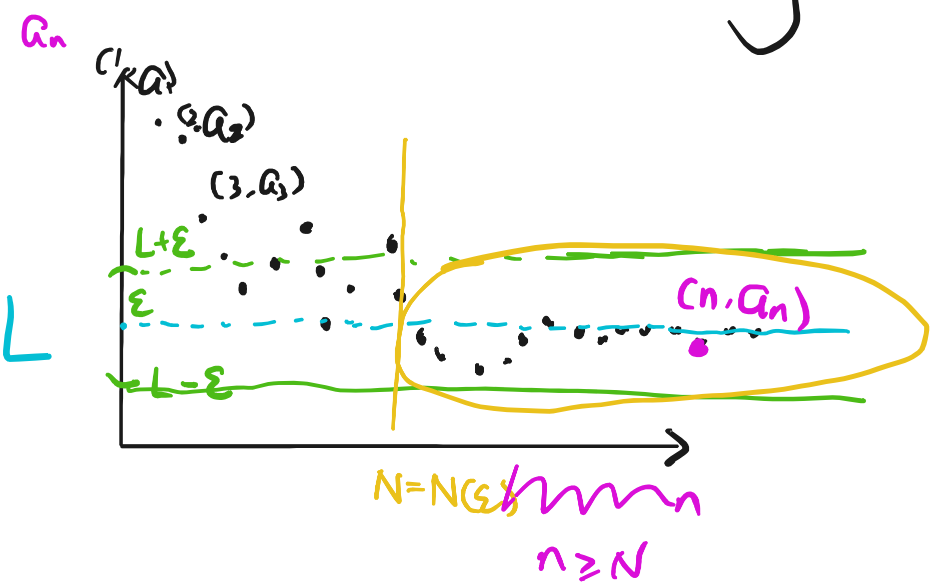
$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \quad (\text{as } n\to\infty)$$

A sequence $(a_n)_{n=1}^{\infty}$ is divergent 發散的

if it is NOT convergent



Example

① $\left(a_n = \frac{1}{n}\right)_{n=1}^{\infty}$

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

pf

we want to find

Given any $\varepsilon > 0$, we want to find

$N = N(\varepsilon)$ $= ?$ if we choose $N = \frac{1}{\varepsilon} + 1$

s.t. = such that 使得 draft

$$\left| \frac{1}{n} - 0 \right| < \varepsilon$$

$$= \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

$\frac{1}{\varepsilon} < \frac{1}{\varepsilon} + 1$

when $n \geq N$.

Formal proof:

For each $\varepsilon > 0$, there exists

$$N = \frac{1}{\varepsilon} + 1$$

s.t.

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

whenever $n \geq N$.

So $\lim_{n \to \infty} \frac{1}{n} = 0$ #

② Assume $|a| < 1$. Prove that

$$\lim_{n\to\infty} a^n = 0.$$

pf <u>case1</u>: $0 < a < 1$

Given any $\varepsilon > 0$, 存在 $\exists$ $N = (\log_a \varepsilon) + 1$

Observe:

① $a^{\log_a \varepsilon} = \varepsilon$

② $a^x < a^y$ if $y < x$

Hope: $n > \log_a \varepsilon$

s.t.

$$|a^n - 0| = a^n \le a^{(\log_a \varepsilon)+1} < a^{\log_a \varepsilon} = \varepsilon$$

when $n \ge N = (\log_a \varepsilon) + 1$

<u>case2</u>: $a = 0 \Rightarrow a^n = 0$

$\forall \varepsilon > 0$ $\exists$ $N = 1$ s.t.

$$|a^n - 0| = |0 - 0| = 0 < \varepsilon$$

$\forall$ $n \ge N = 1$

case3: $-1 < a < 0$

__cases:__ $-1 < a < 0$

Note that

$$|a^n - 0| = \begin{cases} a^n = |a|^n & n \text{ is even } \text{偶數} \\ -a^n = |a|^n & n \text{ is odd } \text{奇數} \end{cases}$$

Given $\varepsilon > 0$, $\exists N = (\log_{|a|} \varepsilon) + 1$ s.t.

$$|a^n - 0| = |a|^n \leq |a|^N \stackrel{= a^{(\log_{|a|}\varepsilon)+1}}{<} |a|^{\log_{|a|}\varepsilon} = \varepsilon$$

when $n \geq N = (\log_{|a|} \varepsilon) + 1 > \log_{|a|} \varepsilon$

Therefore, if $|a| < 1$, then

$$\lim_{n \to \infty} a^n = 0$$

#

ξ = xi

③ Assume $|a| < 1$, and $(\xi_n)_{n=1}^{\infty}$ is a sequence s.t. $\xi_n \in \{\pm 1\}$

(i.e. $\xi_n = +1$ or -1).

Prove that

$$\lim_{n\to\infty} (\xi_n \cdot a^n) = 0$$

pf

Note that

$$|\xi_n \cdot a^n - 0| = |\overset{\pm 1}{\overset{\parallel}{\xi_n}} \cdot a^n|$$

$$= \begin{cases} |+a^n| \\ |-a^n| \end{cases} = |a|^n$$

Given $\varepsilon > 0$, $\exists\ N = (\log_{|a|} \varepsilon) + 1$ s.t.

$$|\xi_n \cdot a^n - 0| = |a|^n \leq |a|^{(\log_{|a|}\varepsilon)+1}$$

$$< |a|^{\log_{|a|}\varepsilon} = \varepsilon$$

when $n > N = (\log_{|a|} \varepsilon) + 1$

#

④ Let $a \in \mathbb{R}$, $|a| < 1$.

Let
$$S_n = 1 + a^1 + a^2 + \cdots + a^n$$

Prove that
$$\lim_{n\to\infty} S_n = \frac{1}{1-a}$$

pf
$$S_n = 1 + \cancel{a} + \cancel{a^2} + \cdots + \cancel{a^n}$$
$$-) \quad a \cdot S_n = \cancel{a} + \cancel{a^2} + \cancel{a^3} + \cdots \cancel{a^n} + a^{n+1}$$

$$\underbrace{(1-a)}_{\neq 0 \text{ since } |a|<1} S_n = 1 - a^{n+1}$$

(i.e. = that is
= 也就是)

$$\Rightarrow \quad S_n = \frac{1 - a^{n+1}}{1-a} = \frac{1}{1-a} + \frac{-a^{n+1}}{1-a}$$

By Example ③, $\lim_{n\to\infty} (-a^n) = 0$ ($\xi_n = -1 \ \forall n$)

Consider $(1-a) \cdot \varepsilon > 0$

So $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t.

$N((1-a)\varepsilon)$

$$|-a^n - 0| < \varepsilon$$

$\forall \ n \geq N$.

NOTE:

$|a| < 1$

$\Rightarrow -1 < a < 1$

$\Rightarrow 1 - a > 0$

$\Rightarrow$ When $n \geq N((1-a)\varepsilon)$,

$|-a^n - 0|$

$$\left| -a^{n+1} - 0 \right| = |a|^{n+1} = |a| \cdot |a|^n < |a|^n < (1-a)\cdot\varepsilon$$

$$\Rightarrow \left| S_n - \frac{1}{1-a} \right| = \left| \cancel{\frac{1}{1-a}} + \frac{-a^{n+1}}{1-a} - \cancel{\frac{1}{1-a}} \right|$$

$$= \frac{1}{\underbrace{1-a}_{>0}} \cdot \left| -a^{n+1} - 0 \right| < \frac{1}{1-a} \cdot (1-a)\cdot\varepsilon$$

$$= \varepsilon$$

when $n \geq N((1-a)\cdot\varepsilon)$

So $\displaystyle\lim_{n\to\infty} S_n = \frac{1}{1-a}$ #

Computation of limits

It is difficult to determine the value of $\displaystyle\lim_{n\to\infty} a_n$ by ε - N arguments.

大定理

So we need some theorems to help us compute $\lim_{n\to\infty} a_n$.

= Proposition =小定理

Prop

If a limit exists, then it is unique: if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.

pf

Assume $L \neq M$. $\Rightarrow$ $L-M>0$ or $M-L>0$

Consider the case $L-M>0$ (the other case is similar.)

① For $\varepsilon = \frac{L-M}{2} > 0$, $\exists\ N = N(\frac{L-M}{2})$ s.t.

$$|a_n - L| < \varepsilon \qquad \lim_{n\to\infty} a_n = L$$

$\forall\ n \geq N$

② For the same $\varepsilon = \frac{L-M}{2} > 0$

$\lim a_n = M$

$\exists\ \tilde{N} = \tilde{N}\left(\frac{L-M}{2}\right)$ s.t. $\lim_{n\to\infty} a_n = M$

$$|a_n - M| < \varepsilon$$

$\forall\ n \geq \tilde{N}$

③ Consider $n_0 = \max\left\{ N\left(\frac{L-M}{2}\right), \tilde{N}\left(\frac{L-M}{2}\right) \right\}$

$\Rightarrow\ n_0 \geq N\left(\frac{L-M}{2}\right)$ and $n_0 \geq \tilde{N}\left(\frac{L-M}{2}\right)$

①+② $\Rightarrow$

- $|a_{n_0} - L| < \frac{L-M}{2} = \varepsilon$

  $\Rightarrow \underbrace{L - \frac{L-M}{2}}_{=\frac{L+M}{2}} < a_{n_0} < L + \frac{L-M}{2}$ by ① $\lim_{n\to\infty} a_n = L$

- $|a_{n_0} - M| < \frac{L-M}{2} = \varepsilon$

  $\Rightarrow M - \frac{L-M}{2} < a_{n_0} < \underbrace{M + \frac{L-M}{2}}_{=\frac{L+M}{2}}$ by ②

$$\Rightarrow \frac{L+M}{2} < a_{n_0} < \frac{L+M}{2}$$

$$\Rightarrow \frac{L+M}{2} < \frac{L+M}{2}$$

Contradiction

($\longrightarrow\leftarrow$)

So
$$L = M \quad \#$$

<u>Thm</u> = Theorem = 定理 (Thm 11.3.7 in Textbook)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences., and $\gamma \in \mathbb{R}$. Then

e.g. $\lim\limits_{n\to\infty} \left(\frac{1}{n} + \left(\frac{1}{2}\right)^n\right) = \lim\limits_{n\to\infty} \frac{1}{n} + \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

(i) $$\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$$

e.g. $\lim\limits_{n\to\infty} \frac{2}{n} = 2 \cdot \lim\limits_{n\to\infty} \frac{1}{n} = 0$

(ii) $$\lim_{n\to\infty}(\gamma \cdot a_n) = \gamma \cdot \left(\lim_{n\to\infty} a_n\right)$$

e.g. $\lim\limits_{n\to\infty} \frac{1}{n \cdot 2^n} = \left(\lim\limits_{n\to\infty} \frac{1}{n}\right)\left(\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n\right) = 0$

(iii) $$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$

(iv) if $\lim\limits_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

$\lim_{n\to\infty}$ ... $\lim_{n\to\infty} b_n$

<u>pf of (iii)</u>

Assume

① $\lim_{n\to\infty} a_n = \alpha$ (alpha)

② $\lim_{n\to\infty} b_n = \beta$ (beta)

That is,

① $\forall \varepsilon > 0$, $\exists\, N = N(\varepsilon)$ s.t.

$$|a_n - \alpha| < \varepsilon \qquad \forall\, n \geq N$$

② $\forall \varepsilon > 0$ $\exists\, K = K(\varepsilon)$ s.t.

$$|b_n - \beta| < \varepsilon \qquad \forall\, n \geq K$$

<u>Observation 1</u>: (Goal $\lim_{n\to\infty} a_n b_n = \alpha \cdot \beta$)

$$|a_n b_n - \alpha \cdot \beta|$$

$$= |a_n b_n - \alpha b_n + \alpha b_n - \alpha \beta|$$

$$= \left| (a_n - \alpha) b_n + \alpha (b_n - \beta) \right|$$

Recall (Triangle inequality)
三角不等式
$|a+b| \le |a| + |b|$

$$\le |(a_n - \alpha) b_n| + |\alpha \cdot (b_n - \beta)|$$

$$= |a_n - \alpha| \cdot |b_n| + |\alpha| \cdot |b_n - \beta|$$

Observation 2:

For $\varepsilon = 1$, $\exists K_1 = K(\varepsilon = 1)$ s.t.

$$|b_n - \beta| < \varepsilon = 1 \quad \forall n \ge K_1$$

$\vee |$ $\quad$ $\angle$

$|b_n| - |\beta|$

Recall (三角不等式)

$|a-b| \ge |a| - |b|$

$|a| = |a-b+b|$
$\le |a-b| + |b|$

$$\Rightarrow \boxed{|b_n| < |\beta| + 1 \quad \forall n \ge K_1}$$

Back to the main argument:

want to prove:
$(\lim_{n\to\infty} a_n b_n = \alpha\beta)$

Given $\varepsilon > 0$,

$$\exists\ \tilde{N} = \max\left\{ N\left(\frac{\varepsilon/2}{|\beta|+2}\right),\ K\left(\frac{\varepsilon/2}{|\alpha|+1}\right),\ K_1 \right\}$$

s.t.

$$|a_n b_n - \alpha\beta|$$

$$\leq |a_n - \alpha| \cdot |b_n| + |\alpha| \cdot |b_n - \beta| \quad \text{(Observation 1)}$$

if $n \geq K_1$

$$\leq \underbrace{|a_n - \alpha|}_{< \frac{\varepsilon/2}{|\beta|+2}} \cdot (|\beta|+1) + |\alpha| \cdot \underbrace{|b_n - \beta|}_{< \frac{\varepsilon/2}{|\alpha|+1}} \quad \text{(Observation 2)}$$

$$< \underbrace{\frac{\varepsilon/2}{|\beta|+2} \cdot (|\beta|+1)}_{< \frac{\varepsilon}{2}} + \underbrace{|\alpha| \cdot \frac{\varepsilon/2}{|\alpha|+1}}_{\frac{\varepsilon}{2}} < \varepsilon$$

$\forall\ n \geq \tilde{N}$

\#