

§ Derived Functor

A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories is called exact if

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is exact in $\mathcal{B}$ for any exact seq.

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in $\mathcal{A}$

e.g. The forgetful functor

$${}_R\text{Mod} \rightarrow \text{Ab}$$

is an exact functor.

If a functor is not exact, then

$$F(\Delta) \xrightarrow{d_1} F(B) \xrightarrow{d_2} F(\Gamma)$$

is only a chain complex.

It generates some homologies $\frac{\ker(d_2)}{\operatorname{im}(d_1)}$

We will study those homologies.

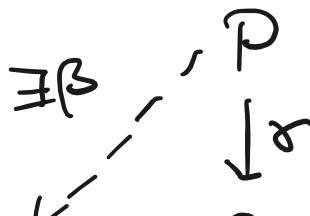
§ Projective resolutions

Def

An object P in an abelian category $\mathcal{A}$ is called projective if it satisfies the following universal property:

Given any surjection $g: B \rightarrow C$ and any morphism $\gamma: P \rightarrow C$

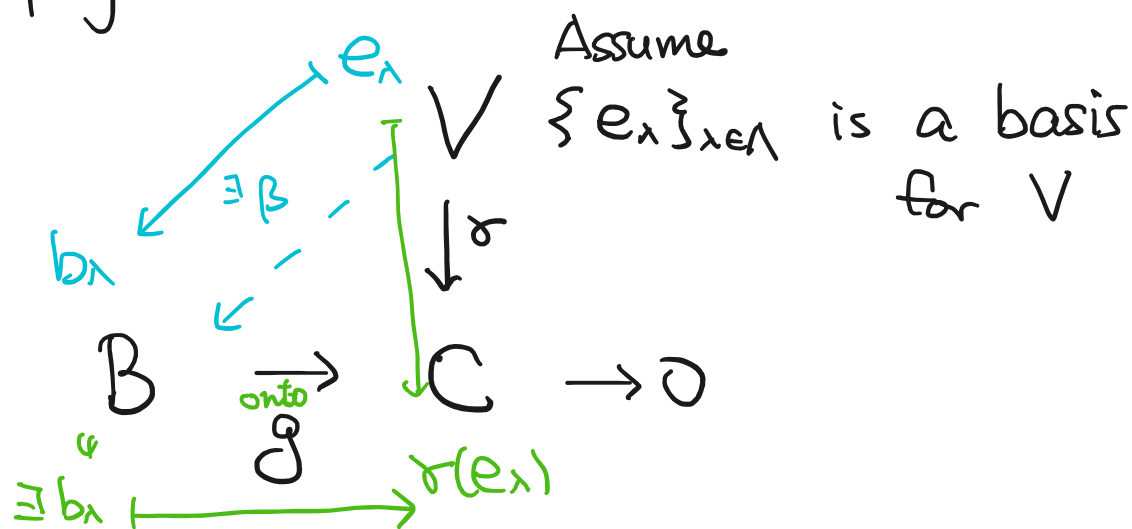
$$\exists \beta: P \rightarrow B \text{ s.t. } \gamma = g \circ \beta$$



$$B \xrightarrow{g} C \rightarrow 0$$

Example

① Any vector space V over k is a projective k -module.



Since g is onto, $\forall \lambda \in \Lambda, \exists b_\lambda \in B$ s.t.

$$g(b_\lambda) = \sigma(e_\lambda)$$

Define $\beta: V \rightarrow B$ by linearity and

$$\beta(e_\lambda) = b_\lambda$$

Then

$$g \circ \beta = \sigma.$$

So V is projective.

② In general, any free R -module is a projective R -module

Proposition

An R -module P is projective iff it is a direct summand of a free R -module, i.e. $\exists$ R -module Q s.t.
 $P \oplus Q$ is a free R -module.

Remark

If $E \rightarrow M$ is a vector bundle, then $\Gamma(E)$ is a projective $C(M)$ -module.

pf of Prop

Assume $P \oplus Q$ is a free R -module

$\cap^{(X)} \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap$

$$K = \text{free } R\text{-mod. with basis } X$$

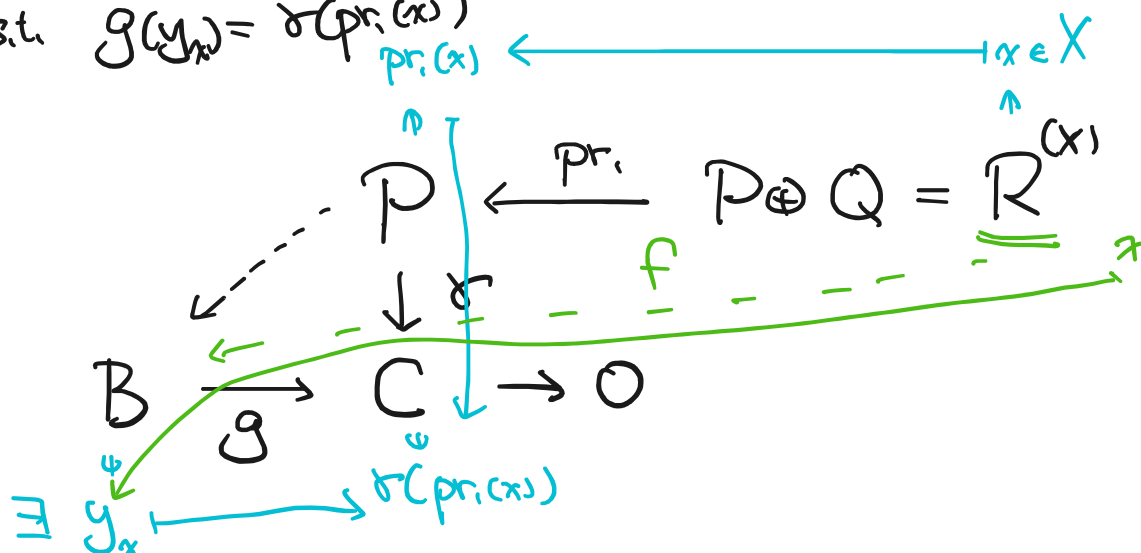
$$= \left\{ \sum_{i=1}^k a_i x_i \mid a_i \in R, x_i \in X \right\}$$

Given a surjection $g: B \rightarrow C$ and

an R -linear map $\delta: P \rightarrow C$,

Consider the R -linear map $f: R^{(X)} \rightarrow B$ st.

$f(x) = y_x \quad \forall x \in X$, where y_x is a chosen element st. $g(y_x) = \delta(\text{pr}_1(x))$



Then

$$g \circ f = \delta \circ \text{pr}_1 : R^{(X)} \rightarrow C$$

Define $\beta: P \rightarrow B$ by

$$P \xrightarrow{i_1} P \oplus Q = R^{(X)} \xrightarrow{f} B$$

β

Then

$$g \circ \beta = g \circ f \circ i_1 = \delta \circ \overbrace{\text{pr}_1 \circ i_1}^{\text{id}_P} = \delta$$

So P is projective.

Conversely, assume P is a projective R -mod.

Consider

$$R^{(P)} = \left\{ \sum_i a_i \cdot x_i \mid a_i \in R, x_i \in P \right\}$$

and the natural map

$$\pi: R^{(P)} \rightarrow P$$

sum in P

$$\pi \left(\sum_i a_i x_i \right) = \sum_i a_i x_i$$

formal sum

which is surjective.

since P is projective

$$\begin{array}{ccc} \textcircled{\exists \iota} & & P \\ \swarrow & \text{---} & \downarrow \text{id}_P \\ R^{(P)} & \xrightarrow{\pi} & P \rightarrow 0 \end{array}$$

Then

$$0 \rightarrow \ker(\pi) \hookrightarrow R^{(P)} \xrightarrow[\pi]{\text{id}} P \rightarrow 0$$

is a split short exact sequence

$$\Rightarrow R^{(P)} \cong P \oplus \ker(\pi) \quad \#$$

Remark

If R is a PID (i.e. a commutative ring without nonzero zero divisors in which every ideal is principal, such as $\mathbb{Z}$),

then it is a theorem that every submodule of a free R -module is free.

$\Rightarrow$ every projective R -module is free

This is NOT true in general.

For example, if $R = \mathbb{Z}/6\mathbb{Z}$, then

$\mathbb{Z}/2\mathbb{Z}$ is a projective R -module since

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$$

This $\mathbb{Z}/2\mathbb{Z}$ is NOT a free $\mathbb{Z}/6\mathbb{Z}$ -module

Remark

A problem of projectives in an arbitrary abelian category is that there might be NO projective objects.

For example, the category of finite abelian groups has NO projective objects.

$$\begin{array}{ccc} & P & \\ \swarrow & \downarrow & \searrow \\ B & \xrightarrow{\beta} C & \rightarrow 0 \end{array}$$

↑
Exercise

An abelian category is said to have enough projectives if for every object A ,

$\exists$ surjection

$$P \rightarrow A$$

with P projective.

The category ${}_R\text{Mod}$ has enough projectives
since we have

$$R^{(A)} \xrightarrow{\pi} A$$

Prop

An R -module P (more generally, an object P in an abelian category) is projective iff the functor

$$\text{Hom}_R(P, -) : {}_R\text{Mod} \rightarrow \text{Ab}$$

is exact.

pf

Suppose $\text{Hom}_R(P, -)$ is exact.

Given any

$$\begin{array}{ccccc} & & P & & \\ & \swarrow h & \downarrow f & & \\ B & \xrightarrow[\text{onto}]{g} & C & \rightarrow & 0 \end{array}$$

$$0 \rightarrow \ker(g) \rightarrow B \xrightarrow{g} C \rightarrow 0 \quad \text{is exact}$$

$$\Rightarrow 0 \rightarrow \text{Hom}_R(P, \ker(g)) \rightarrow \text{Hom}_R(P, B) \xrightarrow{g_*} \text{Hom}_R(P, C) \rightarrow 0$$

is exact

$$\Rightarrow \exists \beta: P \rightarrow B \text{ s.t.}$$

$$g_*(\beta) = g \circ \beta = \gamma$$

So P is projective.

Conversely, suppose that P is projective.

$\forall$ exact

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

Consider

$$0 \rightarrow \text{Hom}_R(P, A) \xrightarrow{f_*} \text{Hom}_R(P, B) \xrightarrow{g_*} \text{Hom}_R(P, C) \rightarrow 0$$

In fact, the exactness at $\text{Hom}_R(P, A)$ and $\text{Hom}_R(P, B)$ always hold (projective assumption is NOT needed), and we'll prove it in the next lemma.

To show g_* is surjective, $\forall \gamma: P \rightarrow C$,

Since $g: B \rightarrow C$ is onto and P is projective,

$\exists \beta: P \rightarrow B$ st

$$\begin{array}{ccc} & \exists \beta & P \\ & \swarrow & \downarrow \sigma \\ B & \xrightarrow{g} & C \rightarrow 0 \end{array}$$

That is,

$$\begin{array}{ccc} \beta & \xrightarrow{\quad} & g_* \beta = \sigma \\ \uparrow & & \uparrow \\ \text{Hom}_R(P, B) & \xrightarrow{g_*} & \text{Hom}_R(P, C) \end{array}$$

g_* is onto.

So the proposition follows from the next lemma:

Lemma

Let $\mathcal{A}$ be an abelian category, and $X \in \text{Ob}(\mathcal{A})$. For any short exact seq

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

in $\mathcal{A}$, the sequences

$$\text{ci) } 0 \rightarrow \text{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{A}}(X, C)$$

$\alpha \mapsto f \circ \alpha$ $\beta \mapsto g \circ \beta$

and

$$(ii) \quad 0 \rightarrow \text{Hom}_A(C, X) \xrightarrow{\gamma^*} \text{Hom}_A(B, X) \xrightarrow{f^*} \text{Hom}_A(A, X)$$

$\gamma \xrightarrow{\quad} \gamma \circ g$
 $\beta \xrightarrow{\quad} \beta \circ f$

are exact sequences in Ab .

pf

We prove it for $A = R\text{Mod}$.

The general case is left as an exercise.

(i) ① $\ker(f_*) = \{0\}$: For $\alpha \in \text{Hom}_A(X, A)$,

$$f_*(\alpha) = f \circ \alpha = 0 \quad \leftarrow \text{zero map}$$

i.e. $\underline{f}(\alpha(x)) = 0 \quad \forall x \in X$

$$\Rightarrow \alpha(x) = 0 \quad \forall x \in X$$

$$\Rightarrow \alpha = 0 \quad \Rightarrow \ker(f_*) = \{0\}$$

② $\text{im}(f_*) = \ker(g_*)$:

(a) $g \circ f = 0 \Rightarrow (g \circ f)_* = g_* \circ f_* = 0$

$$\Rightarrow \text{im}(f_*) \subseteq \ker(g_*)$$

(b) $\ker(g_*) \subseteq \text{im}(f_*)$: Let $\beta \in \ker(g_*) \in \text{Hom}_R(X, B)$

$$\Rightarrow g \circ \beta = 0 \quad \text{i.e.}$$

$$g(\beta(x)) = 0 \quad \forall x \in X$$

$$\beta(x) \in \ker(g) = \text{im}(f) \quad \forall x \in X$$

$$\Rightarrow \exists a_x \in A \text{ s.t. } f(a_x) = \beta(x)$$

Since f is 1-1, such a_x is unique.

Define $\alpha: X \rightarrow A$ by

$$\alpha(x) = a_x$$

Then one can check that $\alpha \in \text{Hom}_A(X, A)$

and

$$f_*(\alpha) = \beta \in \text{im}(f_*)$$

$$\text{So } \ker(g_*) = \text{im}(f_*).$$

$$(ii) \quad (1) \ker(g^*) = \{0\}:$$

$$g^*(\tau) = 0 = \tau \circ g \quad \text{i.e. } \tau(g(b)) = 0 \quad \forall b \in B$$

$$\Rightarrow \tau(\underbrace{g(B)}_C) = 0 \Rightarrow \tau = 0$$

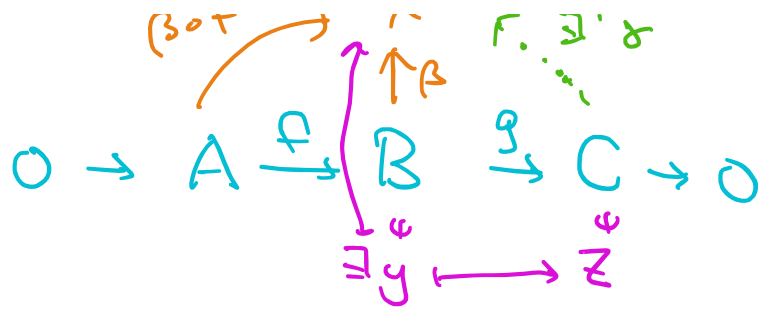
$$(2) \text{im}(g^*) = \ker(f^*):$$

$$(a) \quad g \circ f = 0 \Rightarrow 0 = (g \circ f)^* = f^* \circ g^*$$

$$\Rightarrow \text{im}(g^*) \subseteq \ker(f^*).$$

$$(b) \quad \ker(f^*) \subseteq \text{im}(g^*):$$

$$\text{Let } \beta \in \ker(f^*) \subseteq \text{Hom}_A(B, X) \text{ i.e. } \beta \circ f = 0$$



Define $\delta: C \rightarrow X$ by

$$\delta(z) = \beta(y)$$

where $y \in B$, $g(y) = z$.

Claim: δ is a well-defined R -linear map.
exercise

Assume $y' \in B$, s.t. $g(y') = z = g(y)$

$$\Rightarrow y' - y \in \ker(g) = \operatorname{im}(f)$$

$$\Rightarrow \exists x \in A \text{ s.t. } f(x) = y' - y$$

$$\Rightarrow \beta(y') = \beta(y + f(x)) = \beta(y) + \beta(f(x)) = \beta(y)$$

$\Rightarrow \delta$ is well-defined

Then

$$g^*(x) = \delta \circ g = \beta \in \operatorname{im}(g^*) \quad \#$$

Def

A projective resolution of an object M is a (left) resolution (P, ε) of M , i.e.

$$\dots \xrightarrow{\partial} P_3 \xrightarrow{\partial} P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} M \rightarrow 0$$
is exact, st. each P_i is projective.

Prop

Every R -module M has a projective resolution.

More generally, if an abelian category has enough projectives, then every object in it has a projective resolution.

pf

① Choose a projective P_0 and a surjection $\varepsilon: P_0 \rightarrow M$ and set

$$M_0 = \ker(\varepsilon).$$

② $\exists P_1 \xrightarrow{\varepsilon_1} M_0$, P_1 projective, Set $M_1 = \ker(\varepsilon_1)$;

Inductively, choose a projective P_n and a surjection

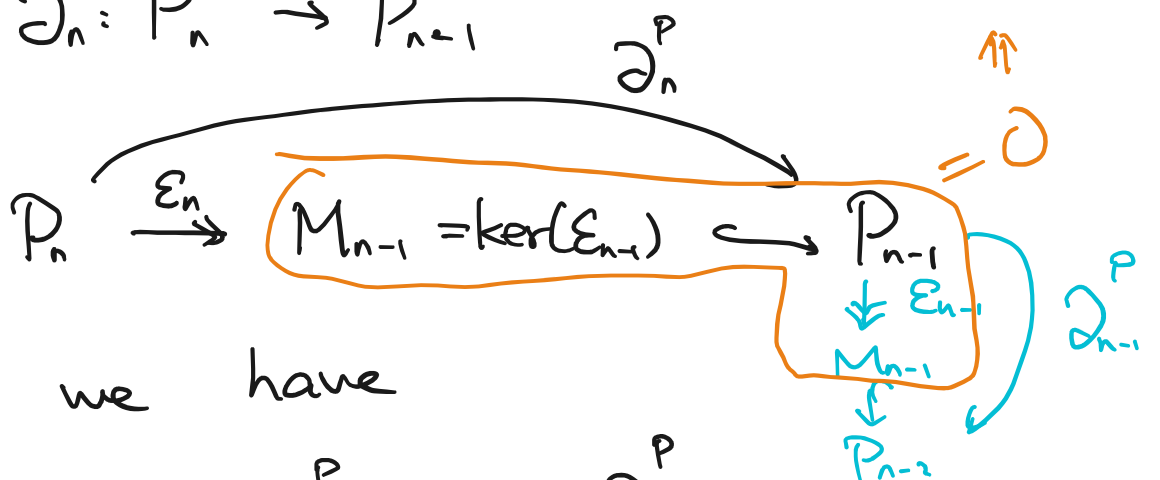
$$E_n: P_n \rightarrow M_{n-1}$$

Set $M_n = \ker(E_n)$

Define

$$\partial_n^P: P_n \rightarrow P_{n-1}$$

by



Then we have

$$\dots \rightarrow P_n \xrightarrow{\partial_n^P} P_{n-1} \xrightarrow{\partial_{n-1}^P} P_{n-2} \rightarrow \dots \rightarrow P_0 \xrightarrow{E_0} M$$

$$\ker(\partial_{n-1}^P) = \ker(E_{n-1}) = \text{im}(E_n)$$

$$\text{im}(\partial_n^P) \xlongequal{\hspace{1.5cm}}$$

#

Thm (Weibel, Comparison Thm 2.2.6)

Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M and $f': M \rightarrow N$.

Then for every resolution $Q_\bullet \xrightarrow{\eta} N$ of N , $\exists$ chain map

$$f: P_\bullet \rightarrow Q_\bullet$$

s.t.

$$\eta \circ f_0 = f' \circ \varepsilon.$$

Such a chain map is unique up to chain homotopy

$$\begin{array}{ccccccc} \dots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 \xrightarrow{\varepsilon} M \rightarrow 0 \\ & & \downarrow \exists f_2 & & \downarrow \exists f_1 & & \downarrow \exists f_0 \\ \dots & \rightarrow & Q_2 & \rightarrow & Q_1 & \rightarrow & Q_0 \xrightarrow{\eta} N \rightarrow 0 \end{array} \quad \begin{array}{c} \downarrow f' \end{array}$$

Remark

two projective resolutions of M

$$\begin{array}{ccccccc} \dots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow M \rightarrow 0 \\ & & \downarrow \text{chain homotopy to } \text{id}_{P_2} & & \downarrow \text{chain homotopy to } \text{id}_{P_1} & & \downarrow \text{id} \\ \dots & \rightarrow & Q_2 & \rightarrow & Q_1 & \rightarrow & Q_0 \rightarrow M \rightarrow 0 \end{array}$$

chain homotopy to id_Q

They form "chain homotopy equivalence"

Next:

injective

$$\begin{array}{ccccc} 0 & \rightarrow & A & \xrightarrow{f} & B \\ & & \alpha \downarrow & \nearrow \exists \beta & \\ & & I & & \end{array}$$