Homological Algebra, week 5, Spring 2025

§ Derived Functor

A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called exact if

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$$

is exact in $\mathcal{B}$ for any exact seq.

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$

e.g. The forgetful functor

$${}_R\mathcal{M}od \longrightarrow \mathcal{A}b$$

is an exact functor.

If a functor is not exact, then

$$\mathcal{F}(A) \xrightarrow{d_1} \mathcal{F}(B) \xrightarrow{d_2} \mathcal{F}(C)$$

$F(A) \to F(B) \to F(C)$

is only a chain complex.

It generates some homologies $\frac{\ker(d_2)}{\operatorname{im}(d_1)}$

We will study those homologies.

§ Projective resolutions

<u>Def</u>

An object P in an abelian category $\mathcal{A}$ is called <u>projective</u> if it satisfies the following universal property:

Given any sujection $g : B \to C$ and any morphism $\gamma : P \to C$

$\exists\ \beta : P \longrightarrow B$ s.t. $\gamma = g \circ \beta$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] \\ {} & {} \end{tikzcd}$$

$$B \xrightarrow[g]{} C \to 0$$

Example

① Any vector space V over $\mathbb{k}$ is a projective $\mathbb{k}$-module.

Assume $\{e_\lambda\}_{\lambda \in \Lambda}$ is a basis for V

$$\begin{tikzcd} & V \arrow[dl, dashed, "\exists \beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "\text{onto}", "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

$e_\lambda \mapsto b_\lambda$

$\exists\, b_\lambda \mapsto \gamma(e_\lambda)$

Since g is onto, $\forall \lambda \in \Lambda$, $\exists\, b_\lambda \in B$ s.t.

$$g(b_\lambda) = \gamma(e_\lambda)$$

Define $\beta : V \to B$ by linearity and

$$\beta(e_\lambda) = b_\lambda$$

Then

$$g \circ \beta = \gamma.$$

So V is projective.

② In general, any free R-module is a projective R-module

Proposition

An R-module P is projective iff it is a direct summad of a free R-module, i.e. $\exists$ R-module Q s.t. $P \oplus Q$ is a free R-module.

Remark

If $E \to M$ is a vector bundle, then $\Gamma(E)$ is a projective $C^\infty(M)$-module.

pf of Prop

Assume $P \oplus Q$ is a free R-module

$R^{(X)}$

$R^{(X)}$ = free R-mod. with basis X

$$= \left\{ \sum_{i=1}^{k} a_i x_i \;\middle|\; a_i \in R,\ x_i \in X \right\}$$

Given a surjection $g: B \to C$ and

an R-linear map $\gamma: P \to C$,

Consider the R-linear map $f: R^{(X)} \to B$ s.t.

$f(x) = y_x \quad \forall x \in X$, where $\underset{}{y_x}^{\in B}$ is a chosen

element s.t. $g(y_x) = \gamma(pr_1(x))$

$$\begin{tikzcd}
 & pr_1(x) & & x \in X \arrow[ll, mapsto] \\
 & P \arrow[d, "\gamma"] \arrow[dl, dashed] & P \oplus Q = R^{(X)} \arrow[l, "pr_1"'] \arrow[dll, dashed, "f"'] & \\
B \arrow[r, "g"'] & C \arrow[r] & 0 & \\
\exists\, y_x \arrow[r, mapsto] & \gamma(pr_1(x)) & &
\end{tikzcd}$$

Then

$$g \circ f = \gamma \circ pr_1 : R^{(X)} \to C$$

Define $\beta: P \to B$ by

$$\begin{tikzcd}
P \arrow[r, hook, "i_1"] \arrow[rr, bend left, "\beta"] & P \oplus Q = R^{(X)} \arrow[r, "f"'] & B
\end{tikzcd}$$

Then

$$g \circ \beta = g \circ f \circ i_1 = \gamma \circ \overbrace{pr_1 \circ i_1}^{id_P} = \gamma$$

So P is projective.

Conversely, assume P is a projective R-mod.
Consider

$$R^{(P)} = \left\{ \sum_i a_i \cdot x_i \;\middle|\; a_i \in R,\ x_i \in P \right\}$$

and the natural map

$$\pi : R^{(P)} \to P$$

sum in P

$$\pi\Big(\sum_i a_i x_i \Big) = \sum_i a_i x_i$$

formal sum

which is surjective.

since P is projective

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\, \iota"'] \arrow[d, "\mathrm{id}_P"] & \\ R^{(P)} \arrow[r, "\pi"] & P \arrow[r] & 0 \end{tikzcd}$$

Then

$$\begin{tikzcd} 0 \arrow[r] & \ker(\pi) \arrow[r, hook] & R^{(P)} \arrow[r, "\pi"'] & P \arrow[r] \arrow[l, bend right, dashed, "id"'] & 0 \end{tikzcd}$$

is a split short exact sequence

$\Rightarrow \quad R^{(P)} \cong P \oplus \ker(\pi)$ #

Remark

If R is a PID (i.e. a commutative ring without nonzero $\overset{x\cdot y=0}{\text{zero divisors}}$ in which every ideal is principal, $\langle x\rangle = \{ r\cdot x \}$ such as $\mathbb{Z}$),

then it is a theorem that every submodule of a free R-module is free

$\Rightarrow$ every projective R-module is free

This is NOT true in general.

For example, if $R = \mathbb{Z}/6\mathbb{Z}$, then $\mathbb{Z}/2\mathbb{Z}$ is a projective R-module since

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$$

This $\mathbb{Z}/2\mathbb{Z}$ is NOT a free $\mathbb{Z}/6\mathbb{Z}$-module

<u>Remark</u>

A problem of projectives in an arbitrary abelian category is that there might be NO projective objects.

For example, the category of finite abelian groups has NO projective objects.

exercise

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "\gamma"] & \\ B \arrow[r, "\beta"'] & C \arrow[r] & 0 \end{tikzcd}$$

An abelian category is said to have enough projectives if for every object A, $\exists$ surjection

$$P \to A$$

with P projective.

The category ${}_R\mathrm{Mod}$ has enough projectives since we have

$$R^{(A)} \xrightarrow{\pi} A$$

<u>Prop</u>

An R-module P (more generally, an object P in an abelian category) is projective iff the functor

$$\mathrm{Hom}_R(P, -) : {}_R\mathrm{Mod} \to \mathcal{Ab}$$

is exact.

<u>pf</u>

Suppose $\mathrm{Hom}_R(P, -)$ is exact.

Given any

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "\text{onto}", "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

$0 \to \ker(g) \to B \xrightarrow{g} C \to 0$ is exact

$$\exists \beta \longmapsto \gamma$$

$\Rightarrow$
$$0 \to \operatorname{Hom}_R(P, \ker(g)) \to \operatorname{Hom}_R(P,B) \xrightarrow{g_*} \operatorname{Hom}_R(P,C) \to 0$$
is exact

$\Rightarrow \exists\, \beta : P \to B$ s.t.
$$g_*(\beta) = g \circ \beta = \gamma$$

So P is projective.

Conversely, suppose that P is projective.

$\forall$ exact
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

Consider

$$\forall \gamma \in$$
$$0 \to \operatorname{Hom}_R(P, A) \xrightarrow{f_*} \operatorname{Hom}_R(P,B) \xrightarrow{g_*} \operatorname{Hom}_R(P,C) \to 0$$

In fact, the exactness at $\operatorname{Hom}_R(P,A)$ and $\operatorname{Hom}_R(P,B)$ always hold (projective assumption is NOT needed), and we'll prove it in the next lemma.

To show g_* is surjective, $\forall\, \gamma : P \to C$, since $g: B \to C$ is onto and P is projective

$\exists\ \beta : P \to B$ s.t.

$$\begin{tikzcd} & P \arrow[d, "\gamma"] \arrow[dl, dashed, "\exists \beta"'] \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

That is,

$$\begin{array}{ccc} \beta & \longmapsto & g_*(\beta) = g\circ\beta = \gamma \\ \mathrel{\rotatebox{90}{$\in$}} & & \uparrow \\ \mathrm{Hom}_R(P,B) & \xrightarrow{g_*} & \mathrm{Hom}_R(P,C) \end{array}$$

g_* is onto.

So the proposition follows from the next lemma:

Lemma

Let $\mathcal{A}$ be an abelian category, and $X \in Ob(\mathcal{A})$. For any short exact seq.

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathcal{A}$, the sequences

$$\text{(i)} \quad 0 \to \mathrm{Hom}_{\mathcal{A}}(X,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X,C)$$

$$\alpha \longmapsto f\circ\alpha \qquad \beta \longmapsto g\circ\beta$$

and

(ii) $0 \to \mathrm{Hom}_{\mathcal{A}}(C,X) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(B,X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A,X)$

$\gamma \longmapsto \gamma\circ g \qquad \beta \longmapsto \beta \circ f$

are exact sequences in $\mathcal{Ab}$.

pf

We prove it for $\mathcal{A} = {}_R\mathcal{Mod}$.

The general case is left as an exercise.

(i) ① $\ker(f_*) = \{0\}$: For $\alpha \in \mathrm{Hom}_{\mathcal{A}}(X,A)$,

$$f_*(\alpha) = f\circ\alpha = 0 \leftarrow \text{zero map}$$

i.e. $\quad \overset{1\text{-}1}{f}(\alpha(x)) = 0 \quad \forall\, x \in X$

$\Rightarrow \quad \alpha(x) = 0 \quad \forall x \in X$

$\Rightarrow \quad \alpha = 0 \quad \Rightarrow \ker(f_*) = \{0\}$

② $\mathrm{im}(f_*) = \ker(g_*)$:

(a) $g\circ f = 0 \Rightarrow (g\circ f)_* = g_*\circ f_* = 0$

$\Rightarrow \mathrm{im}(f_*) \subseteq \ker(g_*)$

(b) $\ker(g_*) \subseteq \mathrm{im}(f_*)$: Let $\beta \in \ker(g_*) \subseteq \mathrm{Hom}_R(X,B)$,

$\Rightarrow \quad g\circ\beta = 0 \quad$ i.e.

$g(\beta(x)) = 0 \quad \forall x \in X$

$\beta(x) \in \ker(g) = \mathrm{im}(f) \qquad \forall x \in X$

$\Rightarrow \exists\ a_x \in A$ s.t. $f(a_x) = \beta(x)$

Since f is 1-1, such a_x is unique.

Define $\alpha : X \to A$ by

$$\alpha(x) = a_x$$

Then one can check that $\alpha \in \mathrm{Hom}_R(X, A)$ and

$$f_*(\alpha) = \beta \in \mathrm{im}(f_*)$$

So $\ker(g_*) = \mathrm{im}(f_*)$.

(ii) ① $\ker(g^*) = \{0\}$:

$$g^*(\gamma) = 0 = \gamma \circ g \text{ , i.e. } \gamma(g(b)) = 0 \quad \forall b \in B$$

$$\Rightarrow \gamma\big(\overset{C}{\overset{\parallel}{g(B)}}\big) = 0 \Rightarrow \gamma = 0$$

② $\mathrm{im}(g^*) = \ker(f^*)$:

(a) $g \circ f = 0 \Rightarrow 0 = (g \circ f)^* = f^* \circ g^*$

$\Rightarrow \mathrm{im}(g^*) \subseteq \ker(f^*)$.

(b) $\ker(f^*) \subseteq \mathrm{im}(g^*)$:

Let $\beta \in \ker(f^*) \subseteq \mathrm{Hom}_A(B, X)$ i.e. $\beta \circ f = 0$

$$\begin{tikzcd}
 & & X & & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[ur, bend left, "\beta\circ f"] & B \arrow[r, "g"] \arrow[u, "\beta"] & C \arrow[r] \arrow[ul, dotted, "\exists!\,\gamma"'] & 0 \\
 & & \exists\, y \arrow[r, mapsto] & z &
\end{tikzcd}$$

Define $\gamma : C \to X$ by

$$\gamma(z) = \beta(y)$$

where $y \in B$, $g(y) = z$.

Claim: γ is a well-defined R-linear map.
exercise

Assume $y' \in B$, s.t. $g(y') = z = g(y)$

$\Rightarrow \quad y' - y \in \ker(g) = \operatorname{im}(f)$

$\Rightarrow \exists\, x \in A$ s.t. $f(x) = y' - y$

$\Rightarrow$

$$\beta(y') = \beta(y + f(x)) = \beta(y) + \underbrace{(\beta\circ f)}_{=0}(x)$$
$$= \beta(y)$$

$\Rightarrow \quad \gamma$ is well-defined

Then

$$g^*(\gamma) = \gamma \circ g = \beta \in \operatorname{im}(g^*)$$

\#

Def

A projective resolution of an object M is a (left) resolution $(P_\bullet, \varepsilon)$ of M, i.e.

$$\cdots \xrightarrow{\partial} P_3 \xrightarrow{\partial} P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact, s.t. each P_i is projective.

Prop

Every R-module M has a projective resolution.

More generally, if an abelian category has enough projectives, then every object in it has a projective resolution.

pf

① Choose a projective P_0 and a surjection $\varepsilon: P_0 \to M$ and set

$$M_0 = \ker(\varepsilon_0).$$

② $\exists$ $\underset{\text{projective}}{P_1} \overset{\varepsilon_1}{\twoheadrightarrow} M_0$, Set $M_1 = \ker(\varepsilon_1)$

$\vdots$

Inductively, choose a projective P_n and a surjection

$$\varepsilon_n : P_n \to M_{n-1}$$

Set

$$M_n = \ker(\varepsilon_n)$$

Define

$$\partial_n^p : P_n \to P_{n-1}$$

by

$$\begin{tikzcd} P_n \arrow[r, "\varepsilon_n"] \arrow[rr, bend left, "\partial_n^p"] & M_{n-1} = \ker(\varepsilon_{n-1}) \arrow[r, hook] & P_{n-1} \arrow[d, "\varepsilon_{n-1}"] \arrow[dd, bend left, "\partial^p_{n-1}"] \\ & & M_{n-1} \arrow[d, hook] \\ & & P_{n-2} \end{tikzcd}$$

$= 0$

$\Uparrow$

$\partial^p \circ \partial^p = 0$

Then we have

$$\cdots \to P_n \xrightarrow{\partial_n^p} P_{n-1} \xrightarrow{\partial_{n-1}^p} P_{n-2} \to \cdots \to P_0 \xrightarrow{\varepsilon_0} M$$

$$\ker(\partial_{n-1}^p) = \ker(\varepsilon_{n-1}) = \operatorname{im}(\varepsilon_n)$$

$$\operatorname{im}(\partial_n^p) = \operatorname{im}(\varepsilon_n)$$

#

<u>Thm</u> (Weibel, Comparison Thm 2.2.6)

Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M and $f' : M \to N$.
Then for every resolution $Q_\bullet \xrightarrow{\eta} N$ of N, $\exists$ chain map

$$f : P_\bullet \to Q_\bullet$$

s.t.

$$\eta \circ f_0 = f' \circ \varepsilon .$$

Such a chain map is unique up to chain homotopy

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\exists f_2"'] & P_1 \arrow[r] \arrow[d, dashed, "\exists f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\exists f_0"'] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

Remark

two projective resolutions of M

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, shift right] & P_1 \arrow[r] \arrow[d, dashed, shift right] & P_0 \arrow[r] \arrow[d, dashed, shift right] & M \arrow[r] \arrow[d, shift right, "id"'] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] \arrow[u, dashed, shift right] & Q_1 \arrow[r] \arrow[u, dashed, shift right] & Q_0 \arrow[r] \arrow[u, dashed, shift right] & M \arrow[r] \arrow[u, shift right, "id"'] & 0
\end{tikzcd}$$

chain homotopy to id_P

chain homotopy to id_Q.

They form "chain homotopy equivalence"

Next:

injective

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\exists\beta"] \\ & I & \end{tikzcd}$$