Homological Algebra, week 16, Spring 2025

Special topic: Tamarkin-Tsygan calculi and HKR Thm

Cartan calculus

Let M be a (smooth) mfd. ∃ 3 important operators on differential forms:

- $d_{dR} : \Omega^{\bullet}(M) \longrightarrow \Omega^{\bullet+1}(M)$ de Rham differential/ exterior derivative
- $L_X : \Omega^{\bullet}(M) \longrightarrow \Omega^{\bullet}(M)$ Lie derivative
- $\iota_X : \Omega^{\bullet}(M) \longrightarrow \Omega^{\bullet-1}(M)$ contraction/ interior product

where $X \in \Gamma(T_M)$ is a vector field on M.

They satisfy the following relations:

$$-(-1)^{|\iota_X||d_{dR}|} = -(-1)^{-1\cdot 1}$$

(i) $L_X = \underbrace{[d_{dR}, \iota_X]} = d_{dR} \circ \iota_X + \iota_X \circ d_{dR}$

$[\phi, \psi] = \phi \circ \psi - (-1)^{\overline{|\phi||\psi|}} \psi \circ \phi$ Cartan's formula

(ii) $0 = [d_{dR}, L_X] = d_{dR} \circ L_X - L_X \circ d_{dR}$

(iii) $0 = [d_{dR}, d_{dR}] = d_{dR} \circ d_{dR} + d_{dR} \circ d_{dR}$

(iv) $L_{[X,Y]} = [L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X$

(v) $\iota_{[X,Y]} = [L_X, \iota_Y] = L_X \circ \iota_Y - \iota_Y \circ L_X$

(vi) $0 = [\iota_X, \iota_Y] = \iota_X \circ \iota_Y + \iota_Y \circ \iota_X$

In fact, $[\,,\,]$, ι and L can be extended to polyvector fields:

- $T^{\bullet}_{poly}(M) = \Gamma(\Lambda^{\bullet} T_M) = \bigoplus_{p=0}^{\infty} \Gamma(\Lambda^p T_M)$
- $[\,,\,] : T^p_{poly}(M) \times T^q_{poly}(M) \to T^{p+q-1}_{poly}(M)$,

$[\xi, -]$

(a) $[\xi, \eta \wedge \zeta] = [\xi, \eta] \wedge \zeta + (-1)^{(|\xi|-1)|\eta|} \eta \wedge [\xi, \zeta]$

e.g. $[X, Y \wedge Z] = [X, Y] \wedge Z + Y \wedge [X, Z]$

(b) $[\xi, \eta] = -(-1)^{(|\xi|-1)(|\eta|-1)} [\eta, \xi]$

e.g. $[X \wedge Y, Z] = -(-1)^{1 \cdot 0} [Z, X \wedge Y]$

$= -([Z, X] \wedge Y + X \wedge [Z, Y])$

$[\xi, -]$

(c) $[\xi, [\eta, \zeta]] = [[\xi, \eta], \zeta] + (-1)^{(|\xi|-1)(|\eta|-1)} [\eta, [\xi, \zeta]]$

- $\iota_\xi : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet - |\xi|}(M)$,

  $\iota_{\xi \wedge \eta} = \iota_\xi \circ \iota_\eta$

  e.g. $\iota_{X \wedge Y} = \iota_X \iota_Y$

- $L_\xi : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet - |\xi| + 1}(M)$,

  $L_{\xi \wedge \eta} = (-1)^{|\eta|} L_\xi \circ \iota_\eta + \iota_\xi \circ L_\eta$

  e.g. $L_{X \wedge Y} = -L_X \circ \iota_Y + \iota_X \circ L_Y$

These operators still satisfy (i) – (vi):

Def

A Gerstenhaber algebra is a $\mathbb{Z}$-graded vector sp. $\Theta^\bullet$ $(T_{poly}, \wedge)$ $T_{poly} = \Gamma(\wedge T_M)$

together with

- graded commutative alg. str. on Θ
- graded Lie alg str on $\Theta[1]$

s.t. $[X, Y \cdot Z] = [X, Y] \cdot Z + (-1)^{(|X|-1)|Y|} Y \cdot [X, Z]$

A calculus is a pair (Θ, Ξ) s.t.

(i) Θ is a Gerstenhaber algebra

(ii) Ξ is a graded mod over $(\Theta, \cdot)$ — $\iota_X : \Xi_\bullet \to \Xi_{\bullet + |X|}$

(iii) Ξ is a graded mod over $(\Theta[1], [,])$ — $L_X : \Xi \to \Xi$

(iii) Ξ, is a graded mod ... $\Xi_\bullet \to \Xi_{\bullet+|X|-1}$

(iv) $\exists\ d : \Xi_\bullet \to \Xi_{\bullet-1}$ s.t. $d^2 = 0$

(v) $L_X = [\iota_X, d]$, $\quad \iota_{[X,Y]} = [L_X, \iota_Y]$

Lemma

$$L_{X\cdot Y} = (-1)^{|Y|} L_X \circ \iota_Y + \iota_X \circ L_Y$$

Thm

The pair $\left(\Theta^\bullet = T^\bullet_{poly}(M),\ \Xi_\bullet = \Omega^{-\bullet}(M)\right)$ is a calculus.

Algebraic model (Tamarkin-Tsygan calculus)

Let A be an algebra. For

$$\phi \in C^p(A) = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes p}, A)$$

define

$$\iota_\phi : \underbrace{C_n(A)}_{\overset{\parallel}{A^{\otimes n+1}}} \longrightarrow C_{n-p}(A)$$

$$L_\phi : C_n(A) \longrightarrow C_{n-p+1}(A)$$

by

$$\iota_\phi(a_0 \otimes \cdots \otimes a_n) = a_0\,\phi(a_1, \cdots, a_p) \otimes a_{p+1} \cdots \otimes a_n$$

$$L_\phi(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-p} (-1)^{(p+1)k} a_0 \otimes \cdots a_k \otimes \phi(a_{k+1}, \cdots, a_{k+p}) \otimes \cdots \otimes a_n$$

$$+\sum_{k=n-p+1}^{n} (-1)^{p+1+n(k+1)} \phi(a_{k+1}, \cdots, a_n, a_0 \cdots a_{k+p-n-1}) \otimes a_{k+p-n} \otimes \cdots \otimes a_k$$

Furthermore, define

[Loday, (2.1.7.3)]

Cyclic homology

$$B: C_n(A) \to C_{n+1}(A)$$

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^{nk} (1 \otimes a_k \otimes \cdots a_n \otimes a_0 \cdots \otimes a_{k-1})$$

$$-(-1)^{n+1} \sum_{k=0}^{n} (-1)^{nk} (a_k \otimes 1 \otimes a_{k+1} \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{k-1})$$

<u>Thm</u>

These formulas define operators on Hochschild (co)homology. Furthermore, the pair

$$(\Theta^\bullet, \Xi_\bullet) = (HH^\bullet(A), HH_\bullet(A))$$

with $d = B$ is a calculus.

Relationship between the 2 calculi (HKR)

To obtain precise isomorphisms, we consider a smooth version of Hochschild (co)homology:

$R = C^\infty(M)$

$$C^p_{smooth}(M) = D^p_{poly}(M) = \overbrace{D(M) \otimes_R \cdots \otimes_R D(M)}^{p \text{ times}}$$

$D(M) =$ {diff. op. on M}

$= \{\sum_k X_1^k \circ \cdots \circ X_{i_k}^k\}$

$$C_n^{smooth}(M) = C^\infty(\overbrace{M \times \cdots \times M}^{n+1 \text{ time}})$$

Remark

$x_i^j \in \mathfrak{X}(M)$,

① $$D_{poly}^{p}(M) \subseteq C^{p}(R) = \mathrm{Hom}_{\Bbbk}(R^{\otimes p}, R)$$
$$\phi_1 \otimes \cdots \otimes \phi_p \longmapsto \Phi : f_1 \otimes \cdots \otimes f_p \mapsto \phi_1(f_1) \cdot \phi_2(f_2) \cdots \phi_p(f_p)$$

② $$C^\infty(\overbrace{M \times \cdots \times M}^{n+1}) \cong C^\infty(M) \hat{\otimes}_{\Bbbk} \cdots \hat{\otimes}_{\Bbbk} C^\infty(M)$$
$$\begin{tikzcd} C^\infty(M \times \cdots \times M) \arrow[d, phantom, "\cup" description] & F : (x_0, \cdots, x_n) \mapsto f_0(x_0) \cdots f_n(x_n) \\ C_\bullet(R) = R^{\otimes n+1} \ni & f_0 \otimes \cdots \otimes f_n \arrow[u, mapsto] \end{tikzcd}$$

In fact,

① $(D_{poly}^{\bullet}, d_H)$ is a subcx of $(C^\bullet(R), d_H)$

② b_H $\overset{B,\, L,\, \iota}{\vee}$ on $C_\bullet(R)$ can be extended to $C_\bullet^{smooth}(M)$

$\Rightarrow$ We have a chain cx $(C_\bullet^{smooth}(M), b_H)$

Thm

The pair

$$(\Theta^\bullet, \Xi_\bullet) = \left(H(C^\bullet_{smooth}(M), d_H),\ H(C^{smooth}_{-\bullet}(M), b_H)\right)$$

is a calculus that is isomorphic to the Cartan calculus

$$\left(T_{poly}^{\bullet}(M),\ \Omega^{-\bullet}(M) \right)$$

via the HKR (Hochschild-Kostant-Rosenberg) maps

$$hkr : T_{poly}^{p}(M) \longrightarrow D_{poly}^{p}(M)$$

$$\rotatebox{90}{$\in$}$$

$$X_1 \wedge \cdots \wedge X_p \longmapsto \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{\sigma} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)}$$

$$hkr : C_n^{smooth}(M) \longrightarrow \Omega^n(M)$$

$$f_0 \otimes \cdots \otimes f_n \longmapsto f_0\, d_{dR} f_1 \wedge \cdots d_{dR} f_n$$

Special topic: From Koszul to Fedosov resolutions

Main goal: construct resolutions of $C^\infty(M)$

Koszul contraction

Consider

$$\Omega(M, \hat{S}T_M^*) = \prod_{k=0}^{\infty} \Omega(M, S^k T_M^*)$$

An element in $\Omega(M, \hat{S}T_M^*)$ is locally of the form

$$\sum_{I,J} dx^I \underbrace{a_{I,J}(x)}_{\text{(local) smooth function on } M} \cdot y^J$$

Here $(x^1, \dots, x^n)$ is a local coordinate system on M which induces a local frame $(y^1, \dots, y^n)$ of T_M^*, and $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$,

$$dx^I = (\overbrace{dx^1 \wedge \dots \wedge dx^1}^{i_1 \text{ times}}) \wedge \dots \wedge (\overbrace{dx^n \wedge \dots \wedge dx^n}^{i_n \text{ times}})$$

$$y^J = (\overbrace{y^1 \circ \cdots \circ y^1}^{j_1 \text{ times}}) \circ \cdots \circ (\overbrace{y^n \circ \cdots \circ y^n}^{j_n \text{ times}})$$

NOTE:

x^i and y^i are basically same as local sections of $T_M^* \to M$

Define [illegible]

$$\delta : \Omega^p(M, S^q T_M^*) \to \Omega^{p+1}(M, S^{q-1} T_M^*)$$

$$h : \Omega^p(M, S^q T_M^*) \longrightarrow \Omega^{p-1}(M, S^{q+1} T_M^*)$$

by

$$\delta(dx^I\, a_{I,J}(x)\, y^J) = \sum_{k=1}^{n} dx^k \frac{\partial}{\partial y^k}(dx^I\, a_{IJ}\, y^J)$$

$$= \sum_{k=1}^{n} dx^k \wedge dx^I\, a_{IJ}\, \underbrace{\frac{\partial}{\partial y^k}(y^J)}$$

e.g. $\frac{\partial}{\partial y^1}((y^1)^{\circ 2} \circ y^3)$

$= 2 y^1 \circ y^3$

$$h(dx^I a_{IJ}\, y^J) = \frac{1}{p+q} \sum_{k=1}^{n} \iota_{\frac{\partial}{\partial x^k}}(dx^I)\, a_{IJ}\, y^k \circ y^J$$

<u>Lemma</u>

δ and h are well-defined (indep. of choice of coordinates)

<u>Thm</u>

$$\begin{tikzcd} (C^\infty(M), 0) \arrow[r, shift left, "\iota"] & (\Omega^\bullet(M, \hat{S}T_M^*), \delta) \arrow[l, shift left, "\sigma"] \arrow[loop right, "h"] \end{tikzcd}$$

is a contraction data, where

$$\tau : C^\infty(M) \cong \Omega^0(M, S^0 T_M^*) \hookrightarrow \Omega^\bullet(M, \hat{S} T_M^*)$$

$$\sigma : \Omega^\bullet(M, \hat{S} T_M^*) \twoheadrightarrow \Omega^0(M, S^0 T_M) \cong C^\infty(M)$$

are the trivial injection and projection, respectively

Cor

We have a Koszul resolution

$$C^\infty(M) \xrightarrow{\tau} \Omega^0(M, \hat{S} T_M^*) \xrightarrow{\delta} \Omega^1(M, \hat{S} T_M^*) \xrightarrow{\delta} \cdots,$$

Fedosov contraction

Let

$$\nabla : \Gamma(T_M) \times \Gamma(T_M) \to \Gamma(T_M) \rightsquigarrow \nabla : \Gamma(T_M) \times \Gamma(ST_M) \to \Gamma(ST_M)$$

be an affine connection. Define

$$\overset{\text{pbw}^\nabla}{\overset{\shortparallel}{\text{pbw}}} : \Gamma(ST_M) \longrightarrow D(M)$$

by the iterative formula:

$$\text{pbw}(f) = f \qquad \forall f \in C^\infty(M)$$

$$\text{pbw}(X) = X \qquad \forall X \in \mathfrak{X}(M) = \Gamma(T_M)$$

$$\text{pbw}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \left(\begin{array}{l} X_k \circ \text{pbw}(X_0 \odot \cdots \hat{X}_k \cdots \odot X_n) \\ - \text{pbw}(\nabla_{X_k}(X_0 \odot \cdots \hat{X}_k \cdots \odot X_n)) \end{array} \right)$$

Thm (Laurent-Gengoux, Stiénon, Xu)

$$\mathrm{pbw} : \Gamma(ST_M) \to D(M)$$

is an iso of coalgebras, and it coincides with the infinity jet of the geodesic exponential map

$$\exp^{\nabla} : T_M \to M \times M , \quad v_p \mapsto (p, \exp_p^{\nabla}(v_p))$$

i.e.

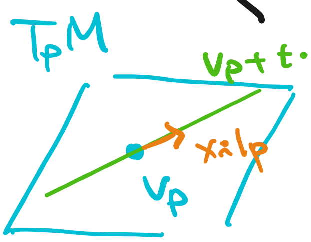
$$\underbrace{\mathrm{pbw}(X_1 \circ \cdots \circ X_n)(f)}_{\in C^\infty(M)}(p) \qquad p \in M \qquad \forall f \in C^\infty(M) \quad X_1, \dots, X_n \in \mathfrak{X}(M)$$

$$T_pM \xrightarrow{\exp_p^{\nabla}} M \xrightarrow{f} \mathbb{R}$$

$$= \left((X_{1,p} \cdots \circ X_{n,p}) \left(\underline{f \circ \exp_p^{\nabla}} \right) \right) (0_p)$$

where $X_{i,p} : C^\infty(T_pM) \to C^\infty(T_pM)$ is the directional derivative along $X_i|_p \in T_pM$:

$$\left(X_{i,p}(g) \right)(v_p) = \left. \frac{d}{dt} \right|_{t=0} g(v_p + t \cdot X_i|_p)$$



Def

$$\nabla^{\varepsilon} : \Gamma(T_M) \times \Gamma(ST_M) \to \Gamma(ST_M)$$

$$\nabla^{\varepsilon}_X(S) := \mathrm{pbw}^{-1}\left(\underbrace{X \cdot \mathrm{pbw}(S)}_{\in D(M)} \right)$$

Lemma

∇^{ε} is a flat connection

In particular, we have the induced covariant derivative

$$(ST_M)^* \\ \shortparallel$$

$$D^{\nabla} := d^{\nabla^{\varepsilon}} : \Omega^{\bullet}(M, \underset{\sim\sim}{\hat{S}T_M^*}) \to \Omega^{\bullet+1}(M, \hat{S}T_M^*)$$

s.t.

$$D^{\nabla} \circ D^{\nabla} = 0.$$

Lemma

Let

Koszul operator ↓

$$\partial := \underline{\underline{\delta}} + D^{\nabla} : \Omega^{\bullet}(M, \hat{S}T_M^*) \to \Omega^{\bullet+1}(M, \hat{S}T_M^*)$$

and

$$F^q = \Omega(M, \hat{S}^{\geq q} T_M^*) = \prod_{k=q}^{\infty} \Omega(M, S^k T_M^*)$$

Then

$$\Omega(M, \hat{S}T_M^*) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots$$

is a complete exhaustive filtration of $\Omega(M, \hat{S}T_M^*)$ s.t.

$$(\partial h)(F^q) \subseteq F^{q+1} \qquad \forall q.$$

In particular, ∂ is a small perturbation of the Koszul contraction

$$⊛ \quad \begin{tikzcd} (C^\infty(M), 0) \arrow[r, shift left, "I"] & (\Omega(M, \hat{S}T_M^*), -\delta) \arrow[l, shift left, "\sigma"] \arrow[loop right, "-h"] \end{tikzcd}$$

See week 14

Now we apply the homological perturbation

lemma to ⊛ we h

lemma [illegible] ⊛, we have:

Thm

∃ a contration data of the form:

$$\begin{tikzcd}(C^\infty(M), 0) \arrow[r, shift left, "I^\nabla"] & (\Omega^\bullet(M, \hat{S}T_M^*), D^\nabla) \arrow[l, shift left, "\sigma"] \arrow[loop right, "h^\nabla"]\end{tikzcd}$$

In particular, we have a Fedosov resolution of $C^\infty(M)$:

$$C^\infty(M) \xrightarrow{I^\nabla} \Omega^0(M, \hat{S}T_M^*) \xrightarrow{D^\nabla} \Omega^1(M, \hat{S}T_M^*) \xrightarrow{D^\nabla} \cdots$$

Remark

① Fedosov resolutions can be constructed for

$$T_{poly}^\bullet(M), \quad D_{poly}^\bullet(M), \quad \underset{\substack{\| \\ \Omega^{-\bullet}(M) \\ \| \\ \text{differential forms}}}{A_\bullet^{poly}(M)}, \quad \underset{\substack{\| \\ \text{another smooth} \\ \text{version of} \\ \text{Hochschild chains}}}{C_\bullet^{poly}(M)}$$

so that the injections I^∇ preserve the calculus structures.

② Fedosov resolutions can be applied to

② Fedosov resolutions can be applied to deformation quantization.

(More precisely, to the globalization of formality morphisms.)