Calculus — Homework 9 (Spring 2025)

1. Let $f(x, y) = x^3 - xy$. Set $\vec{a} = (0, 1)$ and $\vec{b} = (1, 3)$. Find a point $\vec{c}$ on the line segment connecting $\vec{a}$ and $\vec{b}$ for which
$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}).$$

2. Let f be a smooth function on $\mathbb{R}^3$. Show that if $f(\vec{a}) = f(\vec{b})$, then there exists a point $\vec{c}$ between $\vec{a}$ and $\vec{b}$ for which $\nabla f(\vec{c}) \perp (\vec{b} - \vec{a})$.

3. Find the rate of change of f with respect to t along the curve $\vec{\gamma}$.
   (a) $f(x, y) = x^2 y, \quad \vec{\gamma}(t) = e^t\,\vec{\imath} + e^{-t}\,\vec{\jmath}$.
   (b) $f(x, y) = \arctan(y^2 - x^2), \quad \vec{\gamma}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath}$.
   (c) $f(x, y, z) = \ln(x^2 + y^2 + z^2), \quad \vec{\gamma}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + e^{2t}\,\vec{k}$.
   (d) $f(x, y, z) = y \sin(x + z), \quad \vec{\gamma}(t) = 2t\,\vec{\imath} + \cos t\,\vec{\jmath} + t^3\,\vec{k}$.

4. Find $\partial u/\partial s$ and $\partial u/\partial t$.
   (a) $u = x^2 - xy; \quad x = s\cos t,\ y = t\sin s$.
   (b) $u = x^2 \tan y; \quad x = s^2 t,\ y = s + t^2$.
   (c) $u = z^2 \sec(xy); \quad x = 2st,\ y = s - t^2,\ z = s^2 t$.
   (d) $u = xe^{yz^2}; \quad x = \ln(st),\ y = t^3,\ z = s^2 + t^2$.

5. Let
$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$
   Suppose $u = u(x, y)$ is a smooth function.
   (a) Show that
$$\nabla u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e}_\theta,$$
   where $r \neq 0$,
$$\vec{e}_r = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \qquad \text{and} \qquad \vec{e}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}.$$
   (b) Show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

6. Find a normal vector and a tangent vector at the point $\vec{p}$. Write an equation for the tangent line and an equation for the normal line.
   (a) $x^2 + xy + y^2 = 3; \quad \vec{p} = (-1, -1)$.
   (b) $(y - x)^2 = 2x; \quad \vec{p} = (2, 4)$.
   (c) $x^5 + y^5 = 2x^3; \quad \vec{p} = (1, 1)$.

7. Find an equation for the tangent plane at the point $\vec{p}$ and scalar parametric equations for the normal line.
   (a) $z = (x^2 + y^2)^2; \quad \vec{p} = (1, 1, 4)$.
   (b) $xy^2 + 2z^2 = 12; \quad \vec{p} = (1, 2, 2)$.
   (c) $z = \sin x + \sin y + \sin(x + y); \quad \vec{p} = (0, 0, 0)$.

8. Find the point(s) of the surface at which the tangent plane is horizontal.
   (a) $z = 4x + 2y - x^2 + xy - y^2$.
   (b) $z = xy$.