Calculus — Homework 8 (Spring 2025)

1. Calculate the first order and second order partial derivatives.

   (a) $f(x,y) = 3x^2 - xy + y$.

   (b) $f(x,y) = x^2e^{-y}$.

   (c) $f(x,y,z) = z\sin(x-y)$.

   (d) $f(x,y,z) = z^{xy^2}$.

2. Calculate.

   (a) Find $f_x(0,e)$, $f_y(0,e)$, $f_{xy}(0,e)$, $f_{xxx}(0,e)$ and $f_{xyx}(0,e)$ given that $f(x,y) = e^x \ln y$.

   (b) Find $f_x(0,\frac{1}{4}\pi)$, $f_y(0,\frac{1}{4}\pi)$, $f_{yy}(0,\frac{1}{4}\pi)$ and $f_{xyxy}(0,\frac{1}{4}\pi)$ given that $f(x,y) = e^{-x}\sin(x+2y)$.

   (c) Find $f_x(1,2)$, $f_y(1,2)$ and $f_{xx}(1,2)$ given that $f(x,y) = \dfrac{x}{x+y^2}$.

3. Show that the functions u and v satisfy the **Cauchy–Riemann equations**

$$u_x(x,y) = v_y(x,y) \qquad \text{and} \qquad u_y(x,y) = -v_x(x,y).$$

   These equations are fundamentally important in the study of functions of a complex variable.

   (a) $u(x,y) = x^2 - y^2$; $\quad v(x,y) = 2xy$. $\quad$ (Also compute $(x + y\sqrt{-1})^2$.)

   (b) $u(x,y) = e^x \cos y$; $\quad v(x,y) = e^x \sin y$.

   (c) $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$; $\quad v(x,y) = \arctan \dfrac{y}{x}$.

   (d) $u(x,y) = \dfrac{x}{x^2+y^2}$; $\quad v(x,y) = \dfrac{-y}{x^2+y^2}$. $\quad$ (Also compute $\dfrac{1}{(x+y\sqrt{-1})}$.)

4. Let g be a twice differentiable function of one variable and set

$$f(x,y) = g(x+y) + g(x-y).$$

   Show that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

5. Let f be a smooth function of two variables. Show that

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2}.$$

6. Set

$$f(x,y) = \begin{cases} \dfrac{xy(y^2-x^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

   (a) Find the second order partial derivatives of f.

   (b) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

7. Find the gradient.

   (a) $f(x,y) = 3x^2 - xy + y$.

   (b) $f(x,y) = x^2e^{-y}$.

   (c) $f(x,y,z) = z\sin(x-y)$.

   (d) $f(x,y,z) = z^{xy^2}$.

8. Find the gradient at $\vec{p}$.

   (a) $f(x,y) = 2x^2 - 3xy + 4y^2$; $\quad \vec{p} = (2,3)$.

   (b) $f(x,y) = 2x(x-y)^{-1}$; $\quad \vec{p} = (3,1)$.

(c) $f(x,y,z) = e^{-x}\sin(z+2y)$; $\vec{p} = (0, \frac{1}{4}\pi, \frac{1}{4}\pi)$.

(d) $f(x,y,z) = \cos(xyz^2)$; $\vec{p} = (\pi, \frac{1}{4}, -1)$.

9. Find the directional derivative at the point $\vec{p}$ in the direction $\vec{u}$.

(a) $f(x,y) = x^2 + 3y^2$; $\vec{p} = (1,1)$, $\vec{u} = \frac{1}{\sqrt{2}}(1,-1)$.

(b) $f(x,y) = x^2y + \tan y$; $\vec{p} = (-1, \pi/4)$, $\vec{u} = \frac{1}{\sqrt{5}}(1,-2)$.

(c) $f(x,y,z) = xy + yz + zx$; $\vec{p} = (1,-1,1)$, $\vec{u} = \frac{1}{\sqrt{6}}(1,2,1)$.

(d) $f(x,y) = (x + y^2 + z^3)^2$; $\vec{p} = (1,-1,1)$, $\vec{u} = \frac{1}{\sqrt{2}}(1,1,0)$.