Calculus — Homework 7 (Spring 2025)

1. Simplify.

   (a) $(3\vec{a}\cdot\vec{b})-(\vec{a}\cdot 2\vec{b})$.

   (b) $\vec{a}\cdot(\vec{a}-\vec{b})+\vec{b}\cdot(\vec{b}+\vec{a})$.

   (c) $(\vec{a}-\vec{b})\cdot\vec{c}+\vec{b}\cdot(\vec{c}+\vec{a})$.

   (d) $\vec{a}\cdot(\vec{a}+2\vec{c})+(2\vec{b}-\vec{a})\cdot(\vec{a}+2\vec{c})-2\vec{b}\cdot(\vec{a}+2\vec{c})$.

2. Let $\vec{a},\vec{b}\in\mathbb{R}^3$.

   (a) Show that for all vectors $\vec{a}$ and $\vec{b}$

   $$4(\vec{a}\cdot\vec{b})=\|\,\vec{a}+\vec{b}\,\|^2-\|\,\vec{a}-\vec{b}\,\|^2.$$

   (b) Show that $\vec{a}\perp\vec{b}$ iff $\|\,\vec{a}+\vec{b}\,\|=\|\,\vec{a}-\vec{b}\,\|$.

   (c) Show that, if $\vec{a}$ and $\vec{b}$ are nonzero vectors such that

   $$(\vec{a}+\vec{b})\perp(\vec{a}-\vec{b}) \qquad \text{and} \qquad \|\,\vec{a}+\vec{b}\,\|=\|\,\vec{a}-\vec{b}\,\|,$$

   then the parallelogram generated by $\vec{a}$ and $\vec{b}$ is a square.

3. Show that

   $$|\vec{a}\cdot\vec{b}|\le\|\,\vec{a}\,\|\|\,\vec{b}\,\|,$$

   and the equality holds iff $\vec{a}$ and $\vec{b}$ are *parallel*, i.e. there exists λ such that $\vec{a}=\lambda\vec{b}$ or $\vec{b}=\lambda\vec{a}$.

4. Prove the *parallelogram law*:

   $$\|\,\vec{a}+\vec{b}\,\|^2+\|\,\vec{a}-\vec{b}\,\|^2=2\|\,\vec{a}\,\|^2+2\|\,\vec{b}\,\|^2.$$

5. Calculate the $\vec{f}'(t)$ and $\vec{f}''(t)$.

   (a) $\vec{f}(t)=(1+2t)\,\vec{\imath}+(3-t)\,\vec{\jmath}+\cos t\vec{k}$.

   (b) $\vec{f}(t)=e^t(\vec{\imath}-\vec{\jmath})+e^{-2t}(\vec{\jmath}-\vec{k})$.

   (c) $\vec{f}(t)=\big((t^2\,\vec{\imath}-\vec{\jmath})\cdot(\vec{\imath}-t^2\,\vec{\jmath})\big)\,\vec{\imath}$.

6. Find $\lim\limits_{t\to 0}\vec{f}(t)$ if it exists. Explain why if the limit does not exist.

   (a) $\vec{f}(t)=(1+2t)\,\vec{\imath}+(3-t)\,\vec{\jmath}+\dfrac{t}{|t|}\vec{k}$.

   (b) $\vec{f}(t)=e^t(\vec{\imath}-\vec{\jmath})+e^{-2t}(\vec{\jmath}-\vec{k})$.

   (c) $\vec{f}(t)=\dfrac{\sin t}{2t}\,\vec{\imath}+e^{2t}\,\vec{\jmath}+\dfrac{t^2}{e^t}\,\vec{k}$.

   (d) $\vec{f}(t)=t^2\,\vec{\imath}+\dfrac{1-\cos t}{3t}\,\vec{\jmath}+\dfrac{t}{t+1}\,\vec{k}$.

7. Let $\vec{f}$ be a differentiable vector-valued function. Show that if $\|\,\vec{f}(t)\,\|\neq 0$, then

   $$\frac{d}{dt}(\|\,\vec{f}(t)\,\|)=\frac{\vec{f}(t)\cdot\vec{f}'(t)}{\|\,\vec{f}(t)\,\|}.$$

8. Show that $\|\,\gamma(t)\,\|$ is constant iff $\vec{\gamma}(t)\cdot\vec{\gamma}'(t)=0$ for all t.

9. Find the length of the curve.

   (a) $\vec{\gamma}(t)=t\,\vec{\imath}+\frac{2}{3}t^{3/2}\,\vec{\jmath}$, from $t=0$ to $t=8$.

   (b) $\vec{\gamma}(t)=e^t\big(\cos t\,\vec{\imath}+\sin t\,\vec{\jmath}\big)$, from $t=0$ to $t=\pi$.

   (c) $\vec{\gamma}(t)=t\,\vec{\imath}+\ln(\sec t)\,\vec{\jmath}+3\,\vec{k}$, from $t=0$ to $t=\frac{\pi}{4}$.

(d) $\vec{\gamma}(t) = (t\sin t + \cos t)\,\vec{\imath} + (\sin t - t\cos t)\,\vec{\jmath} + \frac{1}{2}\sqrt{3}\,t^2\,\vec{k}$, from $t = 0$ to $t = 2\pi$.

10. Let

$$\vec{\gamma}(t) = 3\cos t\,\vec{\imath} + 3\sin t\,\vec{\jmath} + 4t\,\vec{k}, \qquad t \geq 0.$$

(a) Let

$$s(t) = \int_0^t \| \vec{\gamma}'(u) \| \, du.$$

Show that s is a one-to-one function, and find its inverse function $\tau(s)$.

(b) Let

$$\vec{R}(s) = \vec{\gamma}(\tau(s)).$$

Show that

$$\| \frac{d\vec{R}}{ds} \| = 1.$$

(The parametrization s is called the **parametrization by arc length**.)

(c) Find the length of the curve

$$\vec{R}(s), \qquad 0 \leq s \leq L.$$