Calculus — Homework 6 (Spring 2025)

1. Let α be an arbitrary real number (not necessarily an integer). Set
$$f(x) = (1+x)^\alpha.$$
   (a) Show that the Taylor series of $f(x)$ can be written as
$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$
   which is called the **binomial series**.

   (b) Show that the binomial series converges absolutely on $(-1,1)$.

   (c) Let
$$\varphi(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k, \qquad \forall x \in (-1,1).$$
   Use term-by-term differentiation to show that
$$\begin{aligned} (1+x)\varphi'(x) &= \alpha\varphi(x), \qquad \forall x \in (-1,1), \\ \varphi(0) &= 1. \end{aligned}$$

   (d) Prove that
$$f(x) = \varphi(x), \qquad \forall x \in (-1,1).$$
   (Hint: Differentiate $\dfrac{\varphi(x)}{(1+x)^\alpha}$.)
   That is,
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$
   for all $x \in (-1,1)$.
   In particular, we have
$$\begin{aligned} (1+x)^2 &= 1 + 2x + x^2, \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3, \end{aligned}$$
   and
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$
   if n is a positive integer.

2. We say that a function $f(x)$ is **entire** if $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for all $x \in \mathbb{R}$. Find all the entire functions that satisfy the equation
$$f''(x) + xf(x) = 0.$$

3. Let $\vec{a} = (0,1,2), \vec{b} = (-1,0,1), \vec{c} = (2,1,0)$. Calculate the following.
   (a) $\vec{a} + \vec{b} + \vec{c}$.
   (b) $2(\vec{a} + \vec{b}) - (\vec{a} + \vec{c})$.
   (c) $2 \cdot \vec{a} + 3 \cdot \vec{b} - 4 \cdot \vec{c} + 5 \cdot \vec{0}$.

4. Consider vectors in $\mathbb{R}^3$.
   (a) Find $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$ such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ but $\vec{b} \neq \vec{c}$.
   (b) Show that if $\vec{u} \cdot \vec{b} = \vec{u} \cdot \vec{c}$ for all *unit vectors* $\vec{u}$ (i.e. $\|\vec{u}\| = 1$), then $\vec{b} = \vec{c}$.