Calculus — Homework 4 (Spring 2025)

1. Does the series absolutely converge, conditionally converge or diverge? Explain why.

   (a) $\sum_{k=1}^{\infty}(-1)^k$.

   (b) $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$.

   (c) $\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}$.

   (d) $\sum_{k=2}^{\infty}(-1)^k\frac{k}{\ln k}$.

   (e) $\sum_{k=1}^{\infty}(-1)^k k\sin(1/k)$.

   (f) $\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{k\sqrt{k}}$.

2. Let f be a function which can be differentiated infinitely many times on $(-1,1)$. The **n-th Taylor polynomial** of $f(x)$ is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

   The **n-th remainder** of $f(x)$ is

$$R_n(x) = f(x) - P_n(x).$$

   (a) Prove that

$$R_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)\cdot(x-t)^2 dt$$

   for each $x \in (-1,1)$.

   (b) Find P_4 for $f(x) = \sqrt{1+x}$.

   (c) Show that if $f(x) = \sqrt{1+x}$, then

$$|R_2(x)| < \frac{\sqrt{2}}{32}, \qquad \forall\, x \in (-1/2, 1/2).$$

3. Prove that, for $x \in (0,1)$,

$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k.$$

4. Expand $g(x)$ in the powers of $x-1$ and specify the values of x for which the expansion is valid.

   (a) $g(x) = 3x^3 - 2x^2 + 4x + 1$.

   (b) $g(x) = x^{-1}$.

   (c) $g(x) = e^{-4x}$.

   (d) $g(x) = \sin \pi x$.

   (e) $g(x) = \cos(\frac{1}{2}\pi x)$.