Calculus — Homework 2 (Spring 2025)

1. State whether the sequence converges and, if it does, find the limit.

   (a) $a_n = 2^{2/n}$.

   (b) $a_n = \left(\frac{2}{n}\right)^n$.

   (c) $a_n = \dfrac{\ln(n+1)}{n}$.

   (d) $a_n = \dfrac{4^{100n}}{n!}$.

   (e) $a_n = \displaystyle\int_{-n}^{0} e^{2x}\,dx$.

   (f) $a_n = n^2 \sin\dfrac{\pi}{n}$.

   (g) $a_n = \left(\dfrac{n-1}{n}\right)^n$.

   (h) $a_n = \displaystyle\int_0^{1/n} \cos e^x\,dx$.

   (i) $a_n = \left(\dfrac{1}{2} + \dfrac{3}{n}\right)^{3n}$.

2. Show that if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$.

3. Prove the following.

   (a) $e^x = o(e^{e^x})$.

   (b) $\ln(\ln x) = o(\ln x)$.

   (c) $3x^5 - 100x^2 + 5x + 1 = O(x^5)$.

   (d) $2^x = O(2^{x^2})$.

4. Evaluate the integrals.

   (a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

   (b) $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

   (c) $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+5x+6}$.

   (d) $\displaystyle\int_{-\infty}^\infty \frac{1}{e^x + e^{-x}}\,dx$.

5. Let f be a continuous function on $(-\infty, \infty)$. Assume that both $\int_0^\infty f(x)\,dx$ and $\int_{-\infty}^0 f(x)\,dx$ converge. Show that, for any real number c, the integrals $\int_c^\infty f(x)\,dx$ and $\int_{-\infty}^c f(x)\,dx$ converge, and

$$\int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

6. This problem shows that $\int_{-\infty}^\infty f(x)\,dx$ and $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$ are different.

   (a) Show that $\displaystyle\int_0^\infty \frac{2x}{x^2+1}\,dx$ diverges and hence that $\displaystyle\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges.

   (b) Show that

$$\lim_{b\to\infty}\int_{-b}^b \frac{2x}{x^2+1}\,dx = 0.$$