Calculus — Homework 1 (Spring 2025)

1. Determine the boundedness and monotonicity (i.e. increasing or decreasing or neither) of the sequence with a_n, as indicated.

   (a) $a_n = \dfrac{2}{n}$.

   (b) $a_n = \dfrac{(-1)^n}{n}$.

   (c) $a_n = \dfrac{n + (-1)^n}{n}$.

   (d) $a_n = \sqrt{n^2+1}$.

   (e) $a_n = \dfrac{2^n}{4^n+1}$.

   (f) $a_n = \dfrac{1}{2n} - \dfrac{1}{2n+3}$.

   (g) $a_n = \ln\left(\dfrac{n+1}{n}\right)$.

   (h) $a_n = \sin\dfrac{\pi}{n+1}$.

   (i) $a_n = \dfrac{3^n}{(n+1)^2}$.

2. Let a_n be the sequence satisfying the given rules. Find an explicit formula for a_n that does not involve an recursive relation. Prove your answer.

   (a) $a_1 = 1;\qquad a_{n+1} = \frac{1}{2}a_n + 1$.

   (b) $a_1 = 1;\qquad a_{n+1} = \dfrac{n}{n+1}a_n$.

   (c) $a_1 = 1;\qquad a_{n+1} = a_n + \dfrac{1}{n(n+1)}$.

   (d) $a_1 = 1;\qquad a_{n+1} = a_n + \cdots + a_1$.

3. Assume $|r| < 1$. Prove that $\lim\limits_{n\to\infty} r^n = 0$.

4. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences. Suppose that $\lim\limits_{n\to\infty} b_n \neq 0$.

   (a) Prove that there exists M such that $b_n \neq 0$ for any $n \geq M$.

   (b) Prove that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

5. Let r be a real number, and
$$a_n = 1 + r + r^2 + \cdots + r^{n-1}.$$

   (a) If $r = 1$, what is a_n for $n = 1, 2, 3, \cdots$?

   (b) If $r \neq 1$, what is a_n for $n = 1, 2, 3, \cdots$? Find a formula for a_n that does not involve adding up the powers of r.

   (c) For what values of r does a_n converge?

   (d) Find the limit $\lim\limits_{n\to\infty} a_n$ for $|r| < 1$.

6. State whether the sequence converges and, if it does, find the limit.

   (a) $a_n = 2^n$.

   (b) $a_n = \dfrac{(-1)^n}{n}$.

   (c) $a_n = \dfrac{n + (-1)^n}{n}$.

   (d) $a_n = \dfrac{4^n}{\sqrt{n^2+1}}$.

   (e) $a_n = \dfrac{2^n}{4^n+1}$.

   (f) $a_n = \dfrac{1}{2n} - \dfrac{1}{2n+3}$.

   (g) $a_n = \ln\left(\dfrac{n+1}{n}\right)$.

   (h) $a_n = \sin\dfrac{\pi}{n+1}$.

   (i) $a_n = \dfrac{(n+1)(n+2)}{(n+3)(n+4)}$.