Calculus — Homework 12 (Spring 2025)

1. Find the Jacobian of the transformation.
   (a) $x = u + v, \quad y = 2u - v.$
   (b) $x = uv, \quad y = u^2 + v^2.$
   (c) $x = (1 + w\cos v)\cos u, \quad y = (1 + w\cos v)\sin u, \quad z = w\sin v.$

2. Calculate the area of the region Ω bounded by the curves

$$x^2 - 2xy + y^2 + x + y = 0, \qquad x + y + 4 = 0.$$

   HINT: Set $u = x - y$, $v = x + y$.

3. Calculate the area of the region Ω bounded by the curves

$$x^2 - 4xy + 4y^2 - 2x - y - 1 = 0, \qquad y = \frac{2}{5}.$$

4. Let T be the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1,$$

   where $a, b, c > 0$.
   (a) Calculate the volume of T by setting

$$x = a\rho\sin\phi\cos\theta, \qquad y = b\rho\sin\phi\sin\theta, \qquad z = c\rho\cos\phi.$$

   (b) Evaluate $\displaystyle\iiint_T \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz.$

5. Evaluate the line integral (i) directly by the definition; and (ii) by applying Green's theorem.
   (a) $\displaystyle\oint_C xy\,dx + x^2\,dy$; where C is the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.
   (b) $\displaystyle\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy$; where C is given by the equation $9x^2 + 4y^2 = 36$.
   (c) $\displaystyle\oint_C y^2\,dx + x^2\,dy$; where C is the boundary of the region that lies between the curves $y = x$ and $y = x^2$.