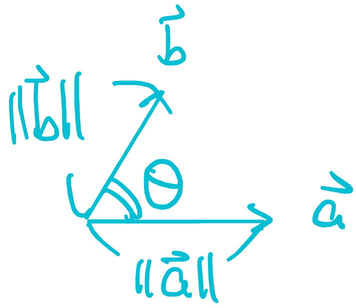
Calculus, week 8, Spring 2025

<u>Recall</u>

$$\vec{a} = (a_1, a_2, a_3) \qquad \in \mathbb{R}^3$$
$$\vec{b} = (b_1, b_2, b_3)$$



$= \sqrt{a_1^2 + \cdots + a_n^2}$

Then

$$\vec{a} \bullet \vec{b} \overset{\text{def}}{=} a_1b_1 + a_2b_2 + a_3b_3$$

$$\underset{\text{Thm}}{=} \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$

easy to generalize:

$$(a_1, \cdots, a_n) \bullet (b_1, \cdots, b_n) = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

<u>NOTE</u>:

① $\theta = \cos^{-1} \dfrac{\vec{a} \bullet \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$

② $\vec{a} \perp \vec{b}$ (i.e. $\vec{a}$ and $\vec{b}$ are

垂直

perpendicular) if $\vec{a} \bullet \vec{b} = 0$

<u>Example</u>

① The vectors $(2,1,1)$ and $(1,1,-3)$

are perpendicular because

$$(2,1,1)\bullet(1,1,-3) = 2\cdot 1 + 1\cdot 1 + 1\cdot(-3)$$
$$= 0 \quad \#$$

② The angle between $(1,1,0)$ and $(0,1,1)$ is

$$\cos^{-1}\left(\frac{(1,1,0)\bullet(0,1,1)}{\|(1,1,0)\|\cdot\|(0,1,1)\|}\right)$$

$$= \cos^{-1}\frac{1\cdot 0 + 1\cdot 1 + 0\cdot 1}{\sqrt{2}\cdot\sqrt{2}}$$

$$= \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \quad \#$$

Thm (§ 13.3)

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$.

(i) $\vec{a}\bullet\vec{a} = \|a\|^2$

$\quad \| \qquad \|$

$a_1\cdot a_1 + a_2\cdot a_2 + a_3\cdot a_3$

(ii) $\vec{a}\bullet\vec{b} = \vec{b}\bullet\vec{a}$

$\quad \| \qquad \|$

$a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$

(iii) $\vec{a}\bullet(\alpha\vec{b} + \beta\vec{c}) = \alpha(\vec{a}\bullet\vec{b}) + \beta(\vec{a}\bullet\vec{c})$

$$= \left(\vec{a}\cdot(\alpha\vec{b})\right) + \left(\vec{a}\cdot(\beta\vec{c})\right)$$

$$a_1(\alpha b_1+\beta c_1)+a_2(\alpha b_2+\beta c_2)+a_3(\alpha b_3+\beta c_3)$$

$$=$$

$$a_1\alpha b_1 + a_1\beta c_1 + a_2\alpha b_2 + a_2\beta c_2 + a_3\alpha b_3 + a_3\beta c_3$$

$$=$$

$$\alpha(\underbrace{a_1b_1+a_2b_2+a_3b_3}_{\vec{a}\cdot\vec{b}}) + \beta(\underbrace{a_1c_1+a_2c_2+a_3c_3}_{\vec{a}\cdot\vec{c}})$$

Example

$$(1,1,0)\cdot\left(2(1,2,3)-(0,0,1)\right)$$

$$= 2\left((1,1,0)\cdot(1,2,3)\right) - \overset{=0}{\cancel{(1,1,0)\cdot(0,0,1)}}$$

$$= 2(1+2+0) = 6 \quad \#$$

Example

Assume

$$\|\vec{a}\|=1,\quad \|\vec{b}\|=3,\quad \|\vec{c}\|=4,$$

$$\vec{a}\cdot\vec{b}=0,\quad \vec{a}\cdot\vec{c}=1,\quad \vec{b}\cdot\vec{c}=-2.$$

Find

① $(3\vec{a})\cdot(\vec{b}+4\vec{c})$

② $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$

③ $\left((\vec{b}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\vec{b}\right)\cdot\vec{c}$

<u>sol</u>

<u>NOTE</u>:

$$(\alpha\vec{a}+\beta\vec{b})\cdot\vec{c} \overset{(ii)}{=} \vec{c}\cdot(\alpha\vec{a}+\beta\vec{b})$$

$$\overset{(iii)}{=} \alpha\,\vec{c}\cdot\vec{a}+\beta\,\vec{c}\cdot\vec{b} \overset{(ii)}{=} \alpha\,\vec{a}\cdot\vec{c}+\beta\,\vec{b}\cdot\vec{c}$$

If $\vec{b}=\vec{0}$

$$(\alpha\vec{a})\cdot\vec{c} = \alpha(\vec{a}\cdot\vec{c})$$

$$= \vec{a}\cdot(\alpha\vec{c})$$

① $(3\vec{a})\cdot(\vec{b}+4\vec{c}) = 3\left(\vec{a}\cdot(\vec{b}+4\vec{c})\right)$

$$\overset{(iii)}{=} 3\underbrace{(\vec{a}\cdot\vec{b})}_{=0}+3\cdot4\,\underbrace{(\vec{a}\cdot\vec{c})}_{=1}$$

$= 12$ #

② $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$

② $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$

$$\overset{(iii)}{=} 2(\vec{a}-\vec{b})\cdot\vec{a} + (\vec{a}-\vec{b})\cdot\vec{b}$$

$$= 2\underbrace{\vec{a}\cdot\vec{a}}_{\overset{(i)}{=}\ \|\vec{a}\|^2=1^2} - 2\underbrace{\vec{b}\cdot\vec{a}}_{\overset{(ii)}{=}\ \vec{a}\cdot\vec{b}=0} + \underbrace{\vec{a}\cdot\vec{b}}_{=0} - \underbrace{\vec{b}\cdot\vec{b}}_{=\|\vec{b}\|^2=9}$$

$$= -7 \quad \#$$

③ $$\big(\underbrace{(\vec{b}\cdot\vec{c})}_{=-2}\vec{a} - \underbrace{(\vec{a}\cdot\vec{c})}_{=1}\vec{b}\big)\cdot\vec{c}$$

$$= -2\cdot\underbrace{\vec{a}\cdot\vec{c}}_{=1} - \underbrace{\vec{b}\cdot\vec{c}}_{=-2} = 0 \quad \#$$

Remark

① All the definitions and properties about vectors can be established for vectors in $\mathbb{R}^2$, and even in $\mathbb{R}^n$ for $n = 1, 2, 3, 4, 5, \ldots$

② Common notations:

$\vec{i} = (1,0,0)$

$\vec{j} =$ [illegible]

$\vec{j} = (0,1,0)$

$\vec{k} = (0,0,1)$

and

$$\vec{a} = (a_1, a_2, a_3)$$
$$= a_1 (1,0,0) + a_2 (0,1,0) + a_3 (0,0,1)$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

§ Limit and vector derivative

Consider

$$\vec{f} : \mathbb{R} \to \mathbb{R}^3$$

Such a function can be expressed as

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$
$$= f_1(t) \cdot \vec{i} + f_2(t) \cdot \vec{j} + f_3(t) \cdot \vec{k}$$

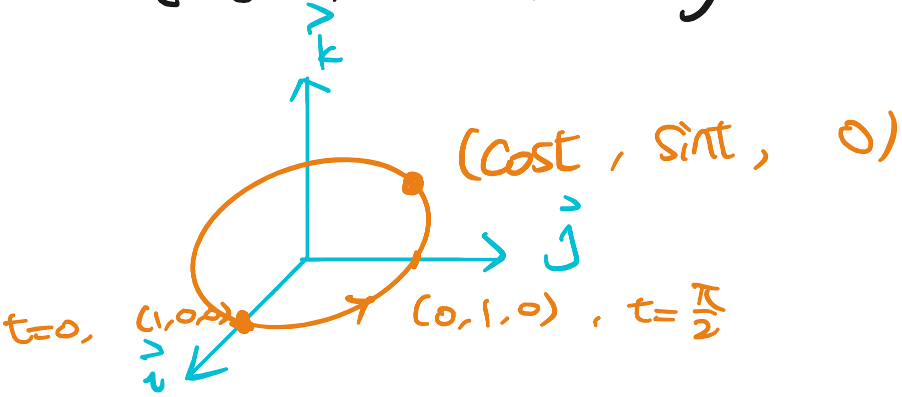
and can be considered as

a curve in $\mathbb{R}^3$.

Example

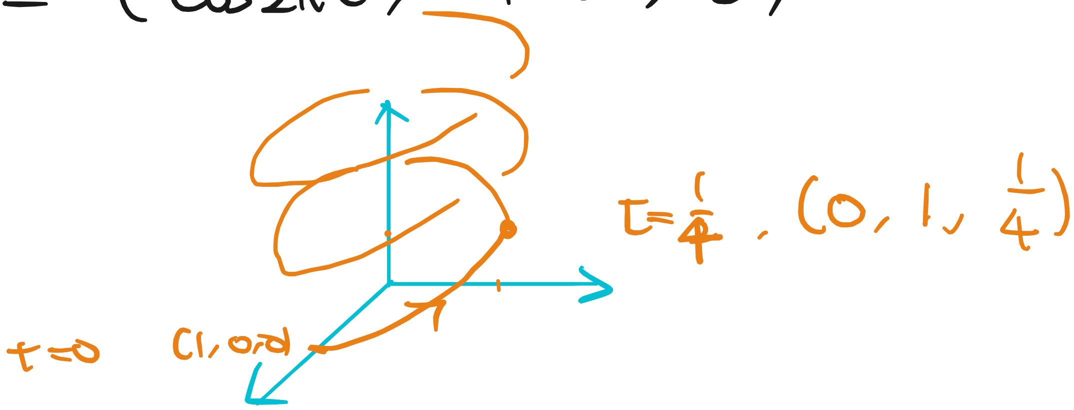
① $\vec{f}(t) = \cos t \cdot \vec{i} + \sin t \cdot \vec{j}$

$= (\cos t, \sin t, 0)$



② $\vec{g}(t) = (\cos 2\pi t)\,\vec{i} + (\sin 2\pi t)\,\vec{j} + t\,\vec{k}$

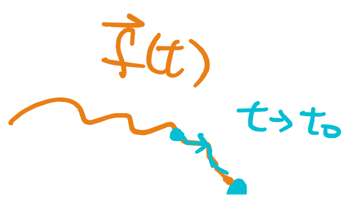
$= (\cos 2\pi t, \sin 2\pi t, t)$



Def (Def 14.1.1)

Let $\vec{f}(t)$ be a function valued in $\mathbb{R}^3$. and $t_0 \in \mathbb{R}$. We say that the limit

$$\lim_{t \to t_0} \vec{f}(t) \quad \text{exists}$$



if $\exists\ L \in \mathbb{K}$ s.t. $\vec{L}$

$$\lim_{t \to t_0} \| \vec{f}(t) - \vec{L} \| = 0$$

Such a vector $\vec{L} \in \mathbb{R}^3$ is called the limit of $\vec{f}(t)$ as $t \to t_0$, denoted

$$\lim_{t \to t_0} \vec{f}(t) = \vec{L}$$

Thm (14.1.4)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ and

$$\vec{L} = (L_1, L_2, L_3) \in \mathbb{R}^3$$

Then

$$\lim_{t \to t_0} \vec{f}(t) = \vec{L}$$

$$\iff \lim_{t \to t_0} f_1(t) = L_1, \quad \lim_{t \to t_0} f_2(t) = L_2,$$

$$\lim_{t \to t_0} f_3(t) = L_3$$

Example

Let

$$\vec{f}(t) = \cos(t+\pi)\,\vec{i} + \sin(t+\pi)\,\vec{j} + e^{-t^2}\,\vec{k}$$

Then

$$\lim_{t\to 0} \vec{f}(t) = \lim_{t\to 0}\left(\cos(t+\pi)\,\vec{i} + \sin(t+\pi)\,\vec{j} + e^{-t^2}\vec{k}\right)$$

$$\overset{\text{Thm}}{=} \underbrace{\left(\lim_{t\to 0}\cos(t+\pi)\right)}_{\cos\pi = -1}\cdot\vec{i} + \underbrace{\left(\lim_{t\to 0}\sin(t+\pi)\right)}_{\sin(\pi)=0}\cdot\vec{j}$$

$$+\underbrace{\left(\lim_{t\to 0} e^{-t^2}\right)}_{e^{-0}=1}\cdot\vec{k}$$

$$= -\vec{i} + \vec{k} = (-1, 0, 1) \qquad \#$$

pf of Thm

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L} \Leftrightarrow \lim_{t\to t_0} \|\vec{f}(t) - \vec{L}\| = 0$$

$$\| \qquad \lim \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2}$$

$$\lim_{t\to t_0} (\ldots) = 0$$

"$\Rightarrow$" Assume $\lim_{t\to t_0} \vec{F}(t) = \vec{L}$, i.e.

Since

$$0 \le |f_1(t) - L_1| = \sqrt{(f_1(t)-L_1)^2}$$

as $t\to t_0$

$$\le \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^3} \to 0$$

by assumption

by the pinching principle.

$$\lim_{t\to t_0} |f_1(t) - L_1| = 0$$

$$\Leftrightarrow \lim_{t\to t_0} f_1(t) = L_1$$

Similarly,

$$\lim_{t\to t_0} f_2(t) = L_2, \quad \lim_{t\to t_0} f_3(t) = L_3$$

"$\Leftarrow$" Assume $\lim_{t\to t_0} f_i(t) = L_i$ $\quad i = 1, 2, 3.$

Then

$$\lim_{t\to t_0} \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2}$$

$$= \sqrt{0^2+0^2+0^2} = 0 \qquad \#$$

<u>Thm</u> (Thm 14.1.3)

Let $\vec{f}$ and $\vec{g}$ be vector-valued functions and u a real-valued function.

Assume

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L} = (L_1, L_2, L_3), \quad \lim_{t\to t_0} \vec{g}(t) = \vec{M} = (M_1, M_2, M_3)$$

$$\lim_{t\to t_0} u(t) = A \in \mathbb{R} \qquad \alpha, \beta \in \mathbb{R}$$

Then

(i) $\displaystyle\lim_{t\to t_0} (\alpha \vec{f}(t) + \beta\, \vec{g}(t)) = \alpha \cdot \vec{L} + \beta \cdot \vec{M}$ ✓

(ii) $\displaystyle\lim_{t\to t_0} (u(t) \cdot \vec{f}(t)) = A \cdot \vec{L}$

(iii) $\displaystyle\lim_{t\to t_0} \| \vec{f}(t) \| = \| \vec{L} \|$

(iv) $\displaystyle\lim_{t\to t_0} \left(\vec{f}(t) \bullet \vec{g}(t) \right) = \vec{L} \bullet \vec{M}$

pf

Assume $\vec{f}(t) = (\underset{\substack{\uparrow \\ L_1}}{f_1(t)}, \underset{\substack{\uparrow \\ L_2}}{f_2(t)}, \underset{\substack{\uparrow \\ L_3}}{f_3(t)})$ as $t \to t_0$

$\vec{g}(t) = (\underset{\substack{\downarrow \\ M_1}}{g_1(t)}, \underset{\substack{\downarrow \\ M_2}}{g_2(t)}, \underset{\substack{\downarrow \\ M_3}}{g_3(t)})$

(i) $$\lim_{t \to t_0} (\alpha \vec{f}(t) + \beta \vec{g}(t))$$

$$= \lim_{t \to t_0} (\alpha f_1(t) + \beta g_1(t), \alpha f_2(t) + \beta g_2(t), \alpha f_3(t) + \beta g_3(t))$$

by last semester

$$= \alpha \lim_{t \to t_0} f_1(t) + \beta \lim_{t \to t_0} g_1(t)$$

$$= \alpha \cdot L_1 + \beta \cdot M_1$$

by previous Thm

$$= \Big(\lim_{t \to t_0} (\alpha f_1(t) + \beta g_1(t)),$$

$$\alpha L_2 + \beta M_2 = \lim_{t \to t_0} (\alpha f_2(t) + \beta g_2(t)),$$

$$\lim_{t \to t_0} (\alpha f_3(t) + \beta g_3(t)) \Big) = \alpha L_3 + \beta M_3$$

$$= \alpha \cdot \vec{L} + \beta \cdot \vec{M}$$

(iv) $$\lim_{t \to t_0} \vec{f}(t) \bullet \vec{g}(t) = \lim_{t \to t_0} \Big(\underset{\substack{\uparrow \\ L_1}}{f_1(t)} \underset{\substack{\downarrow \\ M_1}}{g_1(t)} + \underset{\substack{\uparrow \\ L_2}}{f_2(t)} \underset{M_2}{g_2(t)} + \underset{\substack{\downarrow \\ L_3}}{f_3(t)} \underset{\substack{\downarrow \\ M_3}}{g_3(t)} \Big)$$

$$= L_1M_1 + L_2M_2 + L_3M_3 = L \bullet M$$

Other equalities can be proved by similar computation #

Def

A vector-valued function $\vec{f}$ is continuous at c if

$$\lim_{t \to c} \vec{f}(t) = \vec{f}(c) = (f_1(c), f_2(c), f_3(c))$$

$$\| \qquad \qquad //$$

$$\left(\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \lim_{t \to c} f_3(t) \right)$$

Remark

$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ is continuous at $t = c$

$\iff$ All the $f_1(t)$, $f_2(t)$, $f_3(t)$ are continuous at $t = c$

Def (Def 14.1.5)

A vector-valued $\vec{f}$ is differentiable at t if

$$\lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \quad \text{exists}$$

$$\|$$

$$\lim_{h\to 0}\left(\frac{f_1(t+h)-f_1(t)}{h}, \frac{f_2(t+h)-f_2(t)}{h}, \frac{f_3(t+h)-f_3(t)}{h}\right)$$

If this limit exists, it is called the derivative of $\vec{f}$ at t, denoted by

$\vec{f}'(t)$ or $\dfrac{d\vec{f}}{dt}$

<u>Thm</u> (p. 697)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$.

Then $\vec{f}$ is differentiable at t $\Leftrightarrow$ f_1, f_2, f_3 are differentiable at t.

In this case,

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

<u>Example</u>

① $\vec{f}(t) = (1, 2, 3)$ $\forall t \in \mathbb{R}$

$$\Rightarrow \vec{f}'(t) = \left(\frac{d1}{dt}, \frac{d2}{dt}, \frac{d3}{dt}\right)$$

$$= (0, 0, 0). \qquad \forall t$$

② $\vec{g}(t) = t\,\vec{i} + t^2\,\vec{j} - e^t\,\vec{k}$

$$= (t, t^2, -e^t)$$

$$\Rightarrow \vec{g}'(t) = (t)'\,\vec{i} + (t^2)'\,\vec{j} - (e^t)'\,\vec{k}$$

$$= \vec{i} + 2t\,\vec{j} - e^t\,\vec{k}$$

$$= (1, 2t, -e^t) \qquad \#$$

<u>Thm</u>

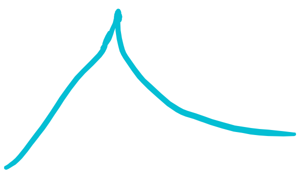
$\vec{f}$ is differentiable at t

$\Rightarrow \vec{f}$ is continuous at t

differentiable curve:

e.g. $f(t) = (t, \sin t, 1)$

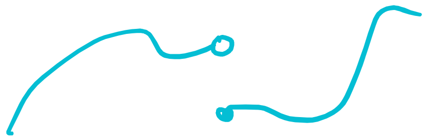
Not differentiable, continuous:



e.g. $f(t) = (t, |t|, t^2)$

Not continuous:

e.g. (t, t^2, t^3) $t<0$



$$\vec{f}(t) = \begin{cases} \ldots & t<0 \\ (\cos t,\ t,\ \sqrt{t}) & t \ge 0 \end{cases}$$

Thm (§14.2)

Let $\vec{f}, \vec{g}$ be differentiable vector-valued functions

u be " real-valued funct

$\alpha, \beta \in \mathbb{R}$.

(i) $\left(\alpha \vec{f}(t) + \beta \vec{g}(t)\right)' = \alpha \vec{f}'(t) + \beta \vec{g}'(t)$

(ii) $(u\vec{f})'(t) = \left(u(t) \cdot \vec{f}(t)\right)'$

$$= u'(t) \cdot \vec{f}(t) + u(t) \cdot \vec{f}'(t)$$

$$\|$$

$$\left(\underbrace{(u(t)\, f_1(t))'}_{u'(t) \cdot f_1(t) + u(t) \cdot f_1'(t)},\ \overset{u'(t) f_2(t) + u(t) f_2'(t)}{\overset{\|}{(u(t)\, f_2(t))'}},\ \underset{u'(t) f_3(t) + u(t) f_3'(t)}{\underset{\|}{(u(t)\, f_3(t))'}} \right)$$

$$= \left(u'(t) f_1(t),\ u'(t) f_2(t),\ u'(t) f_3(t) \right)$$

$$+ \left(u(t) f_1'(t),\ u(t) f_2'(t),\ u(t) f_3'(t) \right)$$

$$= u'(t) \cdot \vec{f}(t) + u(t) \cdot \vec{f}'(t) = \text{RHL}$$

(iii) $(\vec{f}\cdot\vec{g})'(t) = \underline{\vec{f}'(t)\cdot\vec{g}(t)} + \underline{\vec{f}(t)\cdot\vec{g}'(t)}$

$\parallel$

$\big(f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)\big)'$ $\qquad \parallel$

$\parallel$

$\underline{f_1'(t)\,g_1(t)} + f_1(t)\,g_1'(t) + \underline{f_2'(t)\,g_2(t)} + f_2(t)\,g_2'(t)$
$+ \underline{f_3'(t)\,g_3(t)} + f_3(t)\,g_3'(t)$

(iv) $(\vec{f}\circ u)'(t) = \big(\vec{f}(u(t))\big)'$

$\parallel \quad = \cancel{\vec{f}'(u(t))\ u'(t)}$ $\quad \leftarrow$ Chain rule

$u'(t)\cdot\vec{f}'(u(t))$ $\quad =$

$f_2'(u(t))\cdot u'(t)$

$\Big(\big(f_1(u(t))\big)',\ \big(\underline{f_2(u(t))}\big)',\ \big(\underline{f_3(u(t))}\big)'\Big)$

$\parallel$ $\qquad\qquad f_3'(u(t))\cdot u'(t)$

$f_1'(u(t))\cdot u'(t)$

Geometry of curves

"parametrization of a curve"

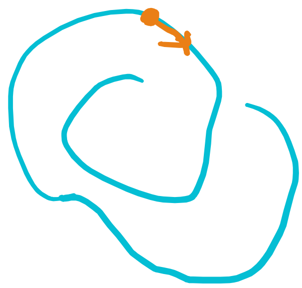
$\downarrow$

$$\underline{\vec{f}(t) = f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}}$$

is a "parametrized curve" in $\mathbb{R}^3$

parameter 参数

parametrize 參數化

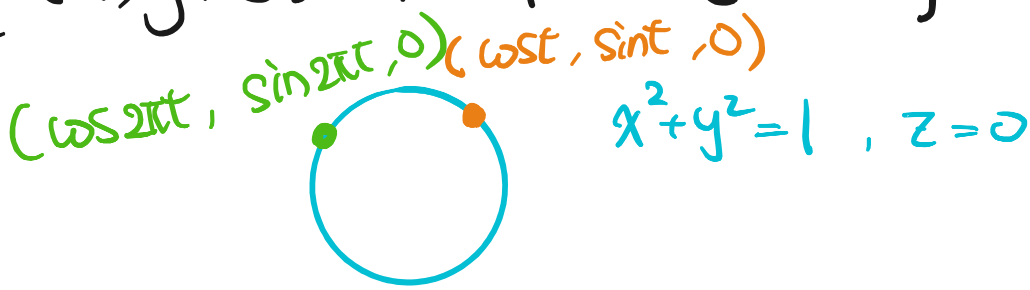


e.g. $\vec{f}(t) = \cos t \ \vec{i} + \sin t \ \vec{j}, \quad t \in \mathbb{R}$

$\vec{g}(t) = \cos 2\pi t \ \vec{i} + \sin 2\pi t \ \vec{j}, \quad t \in \mathbb{R}$

are two parametrizations of the circle

$$\{ (x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \}$$



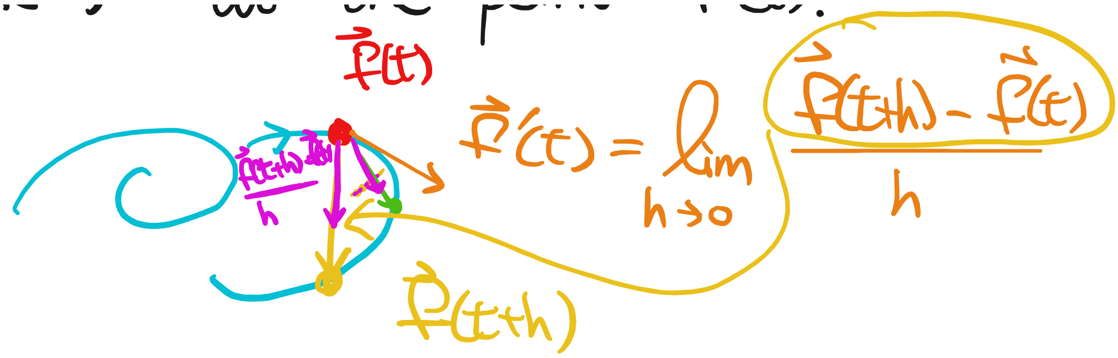
<u>Def</u> (Def. 14.3.1)

Let $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ be a differentiable parametrized curve in $\mathbb{R}^3$.

The vector

$$\vec{f}'(t) = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$$

is called the <u>tangent vector</u> of the curve $\mathrm{im}(\vec{f})$ at the point $\vec{f}(t)$



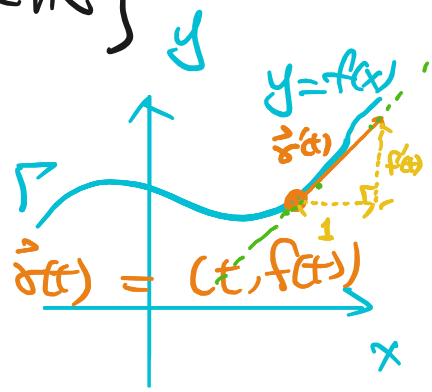
$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Remark

Assume that f is a real-valued differentiable function. Then its graph

$$\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

is a curve in $\mathbb{R}^2$.



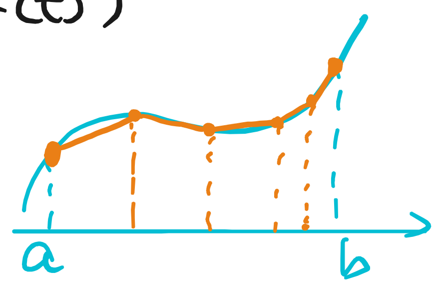
It is parametrized by

$$\vec{r}(t) = (t, f(t)) = t\vec{i} + f(t)\vec{j}$$

$\Rightarrow$

$$\vec{r}\,'(t) = (1, f'(t))$$

Arc length



Recall that the arc length of the curve $y=f(x)$ from a to b is

$$L = \int_a^b \sqrt{1+(f'(x))^2}\; dx$$

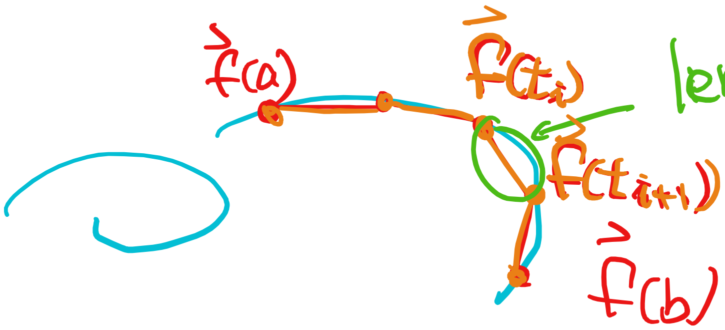
Thm (Thm 14.4.2)

Assume $\vec{f}'(t) \neq 0$

Let $\vec{f}(t)$, $t \in [a,b]$, be a continuously differentiable parametrized curve in $\mathbb{R}^3$. The arc length L of $\vec{f}([a,b])$ is

$$L = \int_a^b \|\vec{f}'(t)\|\; dt$$

$\Delta t = t_{i+1} - t_i$



$$\text{length} = \|\vec{f}(\underbrace{t_{i+1}}_{=\,t_i+\Delta t}) - \vec{f}(t_i)\|$$

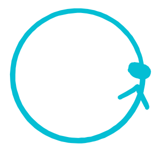
$$= \Delta t \cdot \left\| \frac{\vec{f}(t_i+\Delta t) - \vec{f}(t_i)}{\Delta t} \right\|$$

$$\approx \Delta t \cdot \|\vec{f}'(t_i)\|$$

Example

① $\vec{f}(t) = \cos t\, \vec{i} + \sin t\, \vec{j}$, $t \in [0, 2\pi]$

$\vec{f}'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j}$



$\Rightarrow L = \int^{2\pi} \|\vec{f}'(t)\|\, dt = \int^{2\pi} \sqrt{(\ldots)^2 + \ldots}\, dt$

$$= \int_0 \quad \cdots \quad \int_0 \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \quad \#$$

② $\vec{g}(t) = \cos 2\pi t\, \vec{i} + \sin 2\pi t\, \vec{j}$, $t \in [0,1]$

$$\vec{g}'(t) = -2\pi \sin 2\pi t\, \vec{i} + 2\pi \cos 2\pi t\, \vec{j}$$

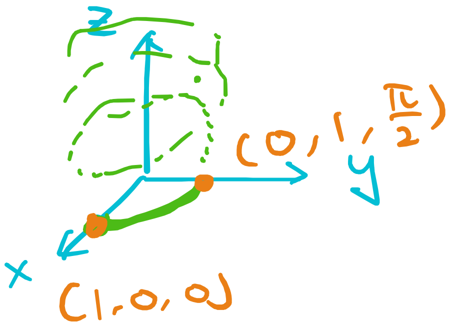
$$\Rightarrow L = \int_0^1 \|\vec{g}'(t)\|\, dt$$

$$= \int_0^1 \sqrt{4\pi^2 (\sin 2\pi t)^2 + 4\pi^2 (\cos 2\pi t)^2}\, dt$$

$$= \int_0^1 2\pi\, dt = 2\pi \quad \#$$

③ $\vec{h}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$,

$t \in [0, \frac{\pi}{2}]$



$$\Rightarrow \vec{h}'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + 1\, \vec{k}$$

$$\Rightarrow L = \int^{\frac{\pi}{2}} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \int_0 \left(\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\right) dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\, dt = \frac{\sqrt{2}}{2}\pi = \sqrt{\frac{\pi^2}{2}}$$

<u>NOTE</u>:

$$\|\vec{h}(0) - \vec{h}(\tfrac{\pi}{2})\| = \sqrt{(1-0)^2 + (0-1)^2 + (0-\tfrac{\pi}{2})^2}$$

$$= \sqrt{2 + \frac{\pi^2}{4}} < \sqrt{\frac{\pi^2}{2}} = L$$

$\frac{\pi^2}{4}$

#

National Tsing Hua University

Calculus (II) – Exam 1

Instructor: Hsuan-Yi Liao

Spring, 2025

Name: ____

Student ID: ____

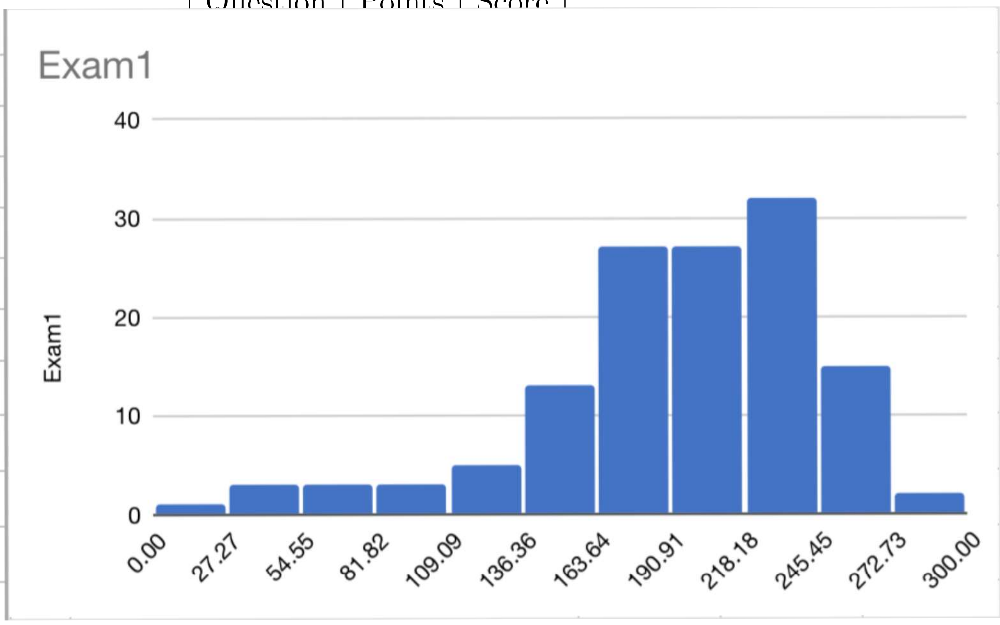
- This exam contains 11 pages (including this cover page) and 9 questions.
- Total of points is 300.
- Time limit: **100 minutes**.

有鑒於還有許多同學對第一次考試範圍的內容不熟悉，我們特別開放一次加分作業讓同學有機會把這部分學得更好。

這次加分作業在eLearn上的“Bonus-Exam1”繳交，收這份作業的時間是從現在到04月17日(週四)23:59。這次作業是要解釋你在Exam1錯在哪裡，並且解釋正確的解法。

Question	Points	Score

Average	193.618
Average (except zero)	195.107
Quartile 0	0
Quartile 1	168.5
Quartile 2	204
Quartile 3	231
Quartile 4	280
標準差	52.7584
非零數	130
不到60%人數	44
滿分人數	0



1. State whether the sequence converges and, if it does, find the limit.

   (a) (10 points) $a_n = \dfrac{1}{2n} - \dfrac{1}{2n+3}$.

   (b) (10 points) $a_n = \left(\dfrac{n-1}{n}\right)^n$.

   (c) (10 points) $a_n = \dfrac{4^{100n}}{n!}$.

   (d) (10 points) $a_n = n^2 \sin \dfrac{\pi}{n}$.

2. (10 points) Prove that $\ln(\ln x) = o(\ln x)$.

3. Does the integral converge or diverge? Explain why.

   (a) (15 points) $\displaystyle\int_0^{\pi/2} \tan x\, dx$.

   (b) (15 points) $\displaystyle\int_\pi^{\infty} \frac{2+\cos x}{x}\, dx$.

4. Evaluate the integrals.

   (a) (15 points) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$.

   (b) (15 points) $\displaystyle\int_{0}^{1} x \ln x\, dx$.

$\sum |a_k|$ converges

$\sum |a_k|$ diverges

$\sum a_k$ converges

5. Does the series absolutely converge, conditionally converge or diverge? Explain why.

(a) (15 points) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$

(b) (15 points) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$.

(c) (15 points) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$.

(d) (15 points) $\displaystyle\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdots 2k}{(2k)!}$.

(e) (15 points) $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{k}{\ln k}$.

(f) (15 points) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{k\sqrt{k}}$.

6. Let f be a function which can be differentiated infinitely many times on $(-1, 1)$. The **n-th Taylor polynomial** of $f(x)$ is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

The **n-th remainder** of $f(x)$ is

$$R_n(x) = f(x) - P_n(x).$$

(a) (15 points) Prove that

$$R_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t) \cdot (x-t)^2 dt$$

for each $x \in (-1, 1)$.

(b) (15 points) Show that if $f(x) = \sqrt{1+x}$, then

$$|R_2(x)| < \frac{\sqrt{2}}{32}, \qquad \forall\, x \in (-1/2, 1/2).$$

7. (20 points) Prove that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

for any real number x.

8. This question aims to show that every bounded sequence has a convergent subsequence.

Let $(a_k)_{k=1}^{\infty}$ be a bounded sequence. Define

$$\bar{a}_n = \sup A_n,$$

where $A_n = \{a_n, a_{n+1}, a_{n+2}, \cdots\}$ and $\sup A_n$ is the least upper bound of the set A_n. In other words, the number $\bar{a}_n$ is characterized by the properties: (i) $\bar{a}_n$ is an upper bound of A_n, i.e. $\bar{a}_n \geq a_k$ for any $k \geq n$; (ii) if M is also an upper bound of A_n, then $\bar{a}_n \leq M$.

(a) (10 points) Prove that the sequence $(\bar{a}_n)_{n=1}^{\infty}$ is a decreasing sequence.

(b) (10 points) Prove that the limit $\lim_{n\to\infty} \bar{a}_n$ exists.

(c) (5 points) Prove that there exists a subsequence $(a_{k_j})_{j=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ such that

$$\lim_{j\to\infty} a_{k_j} = \lim_{n\to\infty} \bar{a}_n.$$

9. A sequence $(a_k)_{k=1}^{\infty}$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists N such that

$$|a_k - a_l| < \varepsilon$$

whenever $k, l \geq N$.

(a) (10 points) Prove that every Cauchy sequence is bounded.

(b) (10 points) Prove that if a Cauchy sequence has a convergent subsequence, then it converges.

(c) (5 points) Prove that every Cauchy sequence converges.

Exam1

1 題目分配

- 1～3: 陳俊碩
- 4～5: 劉筱玫
- 6～9: 潘星翱

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2 Problem 1

State whether the sequence converges and, if it does, find the limit.

(c) $a_n = \frac{4^{100n}}{n!}$

$\sum_{n=1}^{\infty} a_n$

- 本題有不少同學使用 ratio test。注意本題是數列是否收斂而非級數，用 ratio test 證明級數收斂後，要記得說明**因級數收斂，故** $\lim_{n\to\infty} a_n = 0$。

(d) $a_n = n^2 \sin \frac{\pi}{n}$ $\pi n \cdot \dfrac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \to 1$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

- 以下過程是錯誤的：

$$\text{because } -n^2 \le n^2 \sin \frac{\pi}{n} \le n^2 \text{ and } \left(\lim_{n\to\infty} n^2\right) \text{ diverge, } \lim_{n\to\infty} n^2 \sin \frac{\pi}{n} \text{ diverge.}$$

3 Problem 2

Prove that $\ln(\ln x) = o(\ln x)$.

- 計算

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x}$$

時不可以分子分母同取 exp。反例如下：

$$\lim_{x\to\infty} \frac{\ln x}{\ln x^2} = \lim_{x\to\infty} \frac{\ln x}{2\ln x} = \frac{1}{2} \quad \text{但是} \quad \lim_{x\to\infty} \frac{\exp(\ln x)}{\exp(\ln x^2)} = \lim_{x\to\infty} \frac{x}{x^2} = 0$$

$\lim_{a\to\infty} \int_0^a \frac{1}{e^x+e^{-x}} dx$

4 Problem 4

(a) Evaluate the integrals.

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx = \int_{-\infty}^{0} \frac{1}{e^x+e^{-x}} dx + \int_{0}^{\infty} \frac{1}{e^x+e^{-x}} dx$$

$= \lim_{b\to-\infty} \int_b^0 \frac{1}{e^x+e^{-x}} dx$

- 不要算成 Cauchy principal value (雖然在本題答案會一樣)。
- 要分成 $-\infty$ 積到 c 和 c 積到 $+\infty$ 兩個瑕積分。

5 Problem 5

- 數列收斂到 0 **不會**推得級數收斂。
- **絕對收斂**與**條件收斂**的定義不清楚。
- Comparison test 需要注意你所寫的不等式是對所有的 k 都對，還是 k 要大過某個 N。
- Integral test 需要說明數列是遞減且須注意是否是整個數列都遞減，或是當 k 夠大數列才會遞減。
- 注意 Alternative test 的使用前提，**不要**在非交錯級數或非遞減的數列上使用。
- Limit comparison test 需要 $\frac{a_n}{b_n}$ 的極限收斂且非零，極限發散或為 0 時不一定會對。
- Ratio test 和 Root test 都是**極限**小於 1，沒取極限前皆小於 1 是不夠的。
- sin 不恆正。

6 Problem 7

Prove that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

- 泰勒展開式沒有對餘項估計。這樣沒有證明等號成立。