Calculus, week 5, Spring 2025

Recall

Let f be a "smooth" function

i.e. can be differentiated infinitely many times

$f', f'', f''', \dots f^{(n)}, \dots$ exist.

The Taylor expansion of f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(not necessarily converges to $f(x)$)

To get an actual equality, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!} (x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)\,(x-t)^n \, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

for some c (depending x) between x and a.

So

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \iff \lim_{n\to\infty} R_n(x) = 0}$$

§ Power series 冪級數

We consider

$$\sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad \sum_{k=0}^{\infty} a_k (x-a)^k$$

without a given function $f(x)$.

For simplicity, we consider $\sum_{k=0}^{\infty} a_k x^k$.

Parallel results hold for $\sum_{k=0}^{\infty} a_k (x-a)^k$.

Let

A power series $\sum_{k=0}^{\infty} a_k x^k$ converges

(i) at c if $\sum_{k=0}^{\infty} a_k c^k$ converges.

(ii) on the set S if $\sum_{k=0}^{\infty} a_k x^k$ converges at each $x \in S$.

Thm (Thm 12.8.2)

If $\sum_k a_k x^k$ converges at $c \neq 0$, then it converges absolutely at all x with $|x| < |c|$.

If $\sum_k a_k x^k$ diverges at d, then it diverges at all x with $|x| > |d|$.

pf

Suppose $\sum_k a_k c^k$ converges. $(c \neq 0)$

$\Rightarrow \lim a_k c^k = 0$

$$\lim_{k\to\infty} a_k c^k = 0$$

$\Rightarrow$ For k sufficiently large,

$$|a_k \cdot c^k| < 1$$

$$\Rightarrow \quad |a_k \cdot x^k| = \overbrace{|a_k \cdot c^k|}^{<1} \left|\frac{x}{c}\right|^k < \left|\frac{x}{c}\right|^k$$

for k sufficiently large.

So, for $|x| < |c|$,

$$\sum_k \left|\frac{x}{c}\right|^k \quad \text{converges}$$

$$\xRightarrow[\text{test}]{\text{comparison}} \quad \sum_k |a_k x^k| \quad \text{converges}$$

$$\Rightarrow \quad \sum_k a_k x^k \quad \text{converges absolutely}$$

$$\forall \ |x| < |c| \qquad \#$$

By this theorem, there are exactly three prossibilities for convergence of power series:

<u>case 1</u>: $\sum_k a_k x^k$ converges only at $x=0$.

For example, $a_k = k^k$.

$$\sum_k k^k x^k \text{ diverges } \forall x \neq 0$$

since

$$\sqrt[k]{|k^k x^k|} = k \cdot |x| \to \infty > 1 \quad \forall x \neq 0$$

<u>case 2</u>: $\sum_k a_k x^k$ converges absolutely at all real numbers x.

For example, $a_k = \frac{1}{k!}$.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \ (= e^x) \text{ converges absolutely}$$

at any x.

<u>case 3</u>: $\exists\, r > 0$ s.t. $\sum a_k x^k$

converges absolutely for $|x| < r$

diverges for $|x| > r$

For example $a_k = 1$,

$r = 1$

$$\sum_{k=0}^{\infty} x^k \left(= \frac{1}{1-x}\right)$$ converges absolutely for $|x| < 1$

and diverges for $|x| > 1$

<u>Def</u>

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The radius of convergence of a series $\sum_k a_k x^k$ is

(i) 0 if it is case 1

(ii) ∞ if it is case 2

(iii) r if it is case 3

The interval of convergence of $\sum a_k x^k$ is the maximal interval on which it converges.

<u>Example</u>

The radius of convergence

of $\sum k^k x^k$ is 0

① of $\sum_k k^k x^k$ is 0.

② of $\sum_k \frac{x^k}{k!}$ is ∞.

③ of $\sum_k x^k$ is 1.

The interval of convergence

① of $\sum_k k^k x^k$ is $\{0\}$

② of $\sum_k \frac{x^k}{k!}$ is $(-\infty, \infty)$

③ of $\sum_k x^k$ is $(-1, 1)$

$x=1$: $\sum_{k=0}^{\infty} 1^k = 1+1+1+\cdots$ diverges

$x=-1$: $\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+1-1+\cdots$ diverges

<u>Thm</u> (Root test)

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \text{ exists, } = L,$$

then

$$r = \frac{1}{L}$$

is the radius of convergence of

$$\sum_{k=0}^{\infty} a_k x^k .$$

Recall

$\sum b_k$ converges

if $\lim_{k\to\infty} \sqrt[k]{|b_k|} < 1$

and diverges if

$\lim_{k\to\infty} \sqrt[k]{|b_k|} > 1$

pf

Apply Root Test

to

$$\sum_k a_k x^k$$

Consider

$$\lim_{k\to\infty} \sqrt[k]{|a_k x^k|} = \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) \cdot |x| = L \cdot |x|$$

So

$$L \cdot |x| < 1 \overset{\text{Root test}}{\Longrightarrow} \sum |a_k x^k| \text{ converges}$$

$$\Leftrightarrow |x| < \frac{1}{L}$$

$$L \cdot |x| > 1 \overset{\text{Root test}}{\Longrightarrow} \sum |a_k x|^k \text{ diverges}$$

$\Leftrightarrow \quad |x| > \frac{1}{L}$

$\Rightarrow$ Radius of Convergence of $\sum a_k x^k$

$$= \frac{1}{L} \qquad \#$$

e.g.

① $\sum k^k x^k$: $L = \lim_{k\to\infty} \sqrt[k]{k^k} = \lim_{k\to\infty} k = \infty$

$$\Rightarrow \frac{1}{L} = 0$$

③ $\sum x^k$: $L = \lim_{k\to\infty} \sqrt[k]{1} = \lim_{k\to\infty} 1 = 1$

$$\Rightarrow \frac{1}{L} = 1$$

Thm (Ratio test)

If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ exists, $= L$, then

$$r = \frac{1}{L}$$

is the radius of Convergence of

$\sum_k a_k x^k$

Recall

$\sum_k |b_k|$ converges if

$$\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} < 1$$

$\sum_k |b_k|$ diverges if

$$\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} > 1$$

Pf

$$\lim_{k\to\infty} \frac{|a_{k+1} x^{k+1}|}{|a_k x^k|} = \underbrace{\left(\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}\right)}_{=L} |x| = L \cdot |x|$$

So

$L \cdot |x| < 1$ ($|x| < \frac{1}{L}$) $\Rightarrow$ $\sum |a_k x^k|$ converges

$L \cdot |x| > 1$ ($|x| > \frac{1}{L}$) $\Rightarrow$ $\quad$ " $\quad$ diverges

$\Rightarrow$ $\frac{1}{L}$ is the radius of convergence. #

e.g.

② $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$: $\lim_{k\to\infty} \frac{\frac{1}{\cancel{(k+1)!}\ k+1}}{\frac{1}{\cancel{k!}}} = L$

$= \lim_{k\to\infty} \frac{1}{k+1} = 0$

$\Rightarrow$ the radius of convergence of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is ∞

Example

Find the interval of convergence.

Step1: Find the radius of convergence r.

$\sum a_k (x-a)^k$

($\Rightarrow$ $\sum a_k x^k$ converges on $(-r, r)$)

How about $x = \pm r$?

Step2: Check the convergence for $x = \pm r$.

① $\sum_{k=1}^{\infty} \frac{x^k}{k}$:

Step1

$\lim_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = \frac{1}{1} = 1$

$\to 1$

$\Rightarrow$ radius of convergence $= 1$

Step 2:

$x=1$: $\displaystyle\sum_{k=1}^{\infty} \frac{1^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

$x=-1$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ Converges. (alternative series test)

So the interval of convergence is

$[-1, 1)$ #

② $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$:

Step 1

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1}{(\sqrt[k]{k})^2} = \frac{1}{1^2} = 1$$

$\Rightarrow \frac{1}{1} = 1 =$ radius of convergence.

Step 2

$x=1$: $\displaystyle\sum^{\infty} \frac{1}{k^2}$ converges k=1 k converges

$x=-1$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ Converges absolutely

$\Rightarrow$ $[-1, 1]$ is the interval of convergence #

③ $\displaystyle\sum_{k=1}^{\infty} \frac{k}{6^k} \cdot x^k$:

Step1:

$$\lim_{k\to\infty} \frac{\frac{k+1}{\cancel{6^{k+1}}\,6}}{\frac{k}{\cancel{6^k}}} = \lim_{k\to\infty} \frac{1}{6} \cdot \overset{1}{\overset{\uparrow}{\frac{k+1}{k}}} = \frac{1}{6}$$

$\Rightarrow$

$r = \dfrac{1}{\frac{1}{6}} = 6$ is the radius of Convergence.

($\Rightarrow$ it Converges on $(-6, 6)$)

Step2:

$x=6$: $\displaystyle\sum_{k=1}^{\infty} \frac{k}{6^k} \cdot 6^k = \sum_{k=1}^{\infty} k$ diverges

$x=-6$: $\displaystyle\sum_{k=1}^{\infty} \frac{k}{6^k} \cdot (-6)^k = \sum_{k=1}^{\infty} (-1)^k \cdot k$

$= -1+2-3+4-5+6-\cdots$

diverges because

diverges because .....

Consider

$$S_n = \sum_{k=1}^{n} (-1)^k \cdot k$$

NOTE:

$$S_{2n} = \underbrace{(-1+2)}_{+1}\underbrace{(-3+4)}_{+1}\ldots \underbrace{(-(2n-1)+2n)}_{+1}$$

$$= n \quad \text{diverges as } n\to\infty$$

That is $(S_n)_{n=1}^{\infty}$ has a divergent subsequence

$\Rightarrow$ $(S_n)_{n=1}^{\infty}$ is also divergent

$\Rightarrow$ $\sum_{k=1}^{\infty} (-1)^k k$ diverges

Conclusion :

The interval of Convergence is

$(-6, 6)$ #

④ $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} (x+2)^k$:

Step1 :

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k}{k^2 3^k}\right|} = \lim_{k\to\infty} \frac{1}{3}\cdot\left(\frac{1}{\sqrt[k]{k}}\right) = \frac{1}{3}$$

$\sqrt[k]{k} \to 1$

$\Rightarrow r = 3$ is the radius of conv.

$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k$ converges absolutely

$$(-2-3, \; -2+3) = (-5, 1)$$

and it diverges on

$$(-\infty, -5) \cup (1, \infty)$$

Q: Does it converge at $x = -5$ or 1?

<u>Step 2</u>:

$x = -5$:

$$\sum_{k=1}^{\infty} \cancel{\frac{(-1)^k}{k^2 3^k}}\,\cancel{(\cancel{x+2})^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

-3

which is Convergent.

$x = 1$:

$(-1)^k$

$$\sum_{k=1}^{\infty} \cancel{\frac{(-1)^k}{k^2 3^k}}\,\cancel{(\cancel{x+2})^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

3

which is (absolutely) Convergent

So the series $\sum_{k=1} \frac{}{k^2 3^k}(x+2)$

Converges $[-5, 1]$

<u>Thm</u> (Thm 12.9.1 , Thm 12.9.2)

If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ Converges on } (-c, c)$$

(c > 0)

then f is differentiable on $(-c, c)$ and the series

$$\sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) = \sum_{k=1}^{\infty} a_k k \cdot x^{k-1}$$

$$= a_1 + 2a_2 x^1 + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Converges to $f'(x)$ on $(-c, c)$.

In particular, the Taylor expansion of $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is $\sum_{k=0}^{\infty} a_k x^k$.

<u>NOTE:</u>

If $\ell \; \sqrt[n]{|a|} = l$ then

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$, then

$$\lim_{n\to\infty} \sqrt[n]{|a_n\, n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot \underbrace{\sqrt[n]{n}}_{\to 1}$$

$$= L.$$

$\Rightarrow$ In this case,

$$\sum a_k x^k \quad \text{and} \quad \underbrace{\sum_{k=1}^{\infty} a_k \cdot k \cdot x^k}_{x\left(\sum_{k=1}^{\infty} a_k k x^{k-1}\right)} \qquad x^k = x \cdot x^{k-1}$$

have the same radius of convergence

<u>Example</u>

Since

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{on } (-1,1)$$

we have

① $$\left(\frac{1}{1-x}\right)' = \sum_{k=1}^{\infty} k\, x^{k-1} \quad \text{on } (-1,1)$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

② $$\left(\frac{1}{(1-x)^2}\right)' = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$

$$= \sum_{k=1}^{\infty} \left(k x^{k-1}\right)' = \sum_{k=2}^{\infty} k(k-1) x^{k-2}$$

$\forall x \in (-1, 1)$

③ $$\left(\frac{2}{(1-x)^3}\right)' = -6(1-x)^{-4}(-1) = \frac{6}{(1-x)^4}$$

$$= \sum_{k=2}^{\infty} \left(k(k-1)x^{k-2}\right)' = \sum_{k=3}^{\infty} k(k-1)(k-2) x^{k-3}$$

$\forall x \in (-1, 1)$ #

Recall

We considered

$$y'' + y = 0$$

Q: $y = ?$

sol

Assume $y = \sum_{k=0}^{\infty} a_k x^k$

$\Rightarrow$ $y' = \sum^{\infty} k a_k x^{k-1}$

$$k=0$$

$$y'' = \sum_{k=2}^{\infty} k(k-1)\, a_k\, x^{k-2}$$

$n=k-2$

$k=n+2$

$$\Rightarrow 0 = y''+y = \sum_{k=0}^{\infty} a_k x^k + \sum_{k=2}^{\infty} \underbrace{k}_{n+2}\underbrace{(k-1)}_{n+1}\, a_k\, x^{k-2}$$

$$= \sum_{k=0}^{\infty} a_k x^k + \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} \cdot x^n$$

$$= \sum_{k=0}^{\infty} \underbrace{\left(a_k + (k+2)(k+1)\, a_{k+2}\right)}_{=0} x^k = 0$$

$k = m-2 \iff m = k+2$

$$\Rightarrow \underbrace{(k+2)}_{m}\underbrace{(k+1)}_{m-1}\, a_{\underbrace{k+2}_{m}} = -a_{\underbrace{k}_{m-2}} \quad \forall k = 0,1,\dots$$

$$\Rightarrow a_m = \frac{-1}{m(m-1)}\, a_{m-2} \quad \forall m = 2,3,\dots$$

$$\Rightarrow a_{2n} = \frac{-1}{2n(2n-1)}\, a_{2n-2} = \frac{-1}{2n(2n-1)} \cdot \frac{-1}{(2n-2)(2n-3)}\, a_{2n-4}$$

$$= \cdots = \frac{(-1)^n}{(2n)!}\, a_0$$

$$a_{2n+1} = \frac{-1}{(2n+1)(2n)} a_{2n-1} = \frac{-1}{(2n+1)2n} \cdot \frac{-1}{(2n-1)(2n-2)} \cdot a_{2n-3}$$

$$= \cdots = \frac{(-1)^n}{(2n+1)!} a_1$$

$$\Rightarrow y = \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot a_0 \cdot x^{2n}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a_1 \, x^{2n+1}$$

$$= a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad = \cos x$$

$$+ a_1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad = \sin x$$

$$= a_0 \cdot \cos x + a_1 \cdot \sin x$$

#

Thm (Thm 12.9.3)

If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ converges on } (-c, c),$$

then the series

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 x + \frac{a_1}{2} x^2 + \cdots$$

Converges to an antiderivative of $f(x)$ on $(-c, c)$. Equivalently,

$$\int \left(\sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

pf

Assume $\sum_{k=0}^{\infty} a_k x^k$ Converges on $(-c, c)$.

Then $\sum_{k=0}^{\infty} |a_k x^k|$ Converges on $(-c, c)$

Since

$$\left| \frac{a_k}{k+1} x^k \right| \leq |a_k \cdot x^k| \quad \forall k = 0, 1, 2, \cdots$$

by the Comparison Test,

$$\sum \left| \frac{a_k}{k+1} x^k \right| \text{ converges on } (-c,c)$$

$\Rightarrow$

$$x \cdot \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^k = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

also converges on $(-c, c)$

and by the previous thm,

$$\left(\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} \right)' = \sum_{k=0}^{\infty} a_k x^k$$

on $(-c, c)$. #

Example

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{on } (-1, 1)$$

we have

$$\int \frac{1}{1-x} \, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C$$

$$\|$$

$$-\ln(1-x) = \ln\left(\frac{1}{1-x}\right)$$

At $x=0$,

$$\ln\left(\frac{1}{1-0}\right) = 0 = \cancel{\sum_{k=0}^{\infty} \frac{\overset{=0}{x^{k+1}}}{k+1}} + C$$

$\Rightarrow C=0$

$$\Rightarrow \quad \underset{\displaystyle -\ln(1-x)}{\underset{\shortparallel}{\ln\left(\frac{1}{1-x}\right)}} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \forall x \in (-1,1)$$

$$\Rightarrow \quad \ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

For $x \in (-1,1)$, $\underline{\underline{-x}} \in (-1,1)$,

$$\ln(1-(-x)) = -\sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1}$$

$$\boxed{\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}}$$

#

$\forall x \in (-1,1)$

$\forall\ x \in (-1, 1)$

Q: What if $x = \pm 1$?

$\underline{x=-1}$:
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (-1)^{k+1} = -\sum_{k=0}^{\infty} \frac{1}{k+1}$$

which diverges

For $x = 1$, we need the following

$\underline{\text{Thm}}$ (Thm 12.9.5)

Suppose $\underbrace{\sum_{k=0}^{\infty} a_k x^k}_{f(x)=}$ Converges on $(-c, c)$

(i) If f is left continuous at c and $\sum_{k=0}^{\infty} a_k c^k$ Converges, then
$$f(c) = \sum_{k=0}^{\infty} a_k c^k$$

(ii) If f is right continuous at $-c$ and $\sum_{k=0}^{\infty} a_k (-c)^k$ Converges, then
$$f(-c) = \sum_{k=0}^{\infty} a_k (-c)^k$$

Example

We know

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

on $(-1, 1)$, $\ln(1+x)$ is continuous at $x=1$, and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Converges.

So $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \quad \forall\, x \in (-1, 1]$

In particular,

$$\ln 2 = \ln(1+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

#

HW4. 3.

Prove that, for $x \in (0,1)$,

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

by reminders.

<u>pf</u>

$$\ln(1+0) = 0$$

$$\frac{1}{1+x}\Big|_{x=0} = (\ln(1+x))'\Big|_{x=0} = 1$$

$$(\ln(1+x))''\Big|_{x=0} = -(1+x)^{-2}\Big|_{x=0} = -1$$

$$\vdots$$

$$(\ln(1+x))^{(k)}\Big|_{x=0} = (-1)^{k-1}(k-1)!$$

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2!} + \frac{\cancel{2!}\,1}{\cancel{3!}\,3} x^3 - \frac{\cancel{3!}\,1}{\cancel{4!}\,4} x^4 + \cdots$$

$$+ (-1)^{n-1} \frac{\cancel{(n-1)!}\,1}{\cancel{n!}\,n} x^n + R_n(x)$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} x^{k+1} + R_n(x)$$

where

$$R_n(x) = \frac{1}{(n+1)!}\left(\frac{d^{n+1}}{dx^{n+1}}\ln(1+x)\right)\Bigg|_{x=c} \cdot x^{n+1}$$

$$\frac{(-1)^n}{n+1}\cdot(1+c)^{-(n+1)}$$

for some c between 0 and x.

If $x \in (0,1)$, then $c \in (0,x)$

$$\Rightarrow 0 < \frac{x}{1+c} < x$$

$$\Rightarrow |R_n(x)| = \frac{1}{n+1}\left|\frac{x^{n+1}}{(1+c)^{n+1}}\right|$$

$$< \frac{1}{n+1}x^{n+1} \xrightarrow[\because 0<x<1]{} 0$$

as $n \to \infty$

$$\Rightarrow \ln(1+x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1} \quad \forall x \in (0,1)$$

#