Calculus, week 4, Spring 2025

Recall

Consider ($a_k \geq 0$)

$$\sum_{k=1}^{\infty} a_k$$

① If $f(x) > 0$, continuous, decreasing,

$$a_k = f(k),$$

then

$$\sum_{k=1}^{\infty} a_k \text{ Converges} \Leftrightarrow \int_1^{\infty} f(x)\, dx$$

② Comparison test: $a_k \leq b_k \quad \forall k$

$$\sum b_k \text{ conv.} \Rightarrow \sum a_k \text{ Conv.}$$

③ Root test: $\rho = \lim_{k\to\infty} a_k^{\frac{1}{k}}$

$\rho < 1 \Rightarrow \sum a_k$ Conv.

$\rho > 1 \Rightarrow \sum a_k$ div.

④ Ratio test: $\lambda = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$

$\lambda < 1 \Rightarrow \sum a_k$ Conv.

$\lambda > 1 \Rightarrow \Sigma a_k$ div.

Q: What if some $a_k < 0$?

Def

绝對收斂

A series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ converges.

條件收斂

A series is conditionally convergent if it is convergent but NOT absolutely convergent.

Thm (Thm 12.5.1)

Absolutely convergent series are convergent.

pf

Assume $\sum_{k=1}^{\infty} a_k$ is absolutely convergent

$\Rightarrow \sum_{k=1}^{\infty} |a_k|$ converges

$\Rightarrow \quad \sum_{k=1}^{\infty} 2|a_k|$ Converges

Consider

$$\sum_{k=1}^{\infty} (a_k + |a_k|)$$

NOTE: $\quad a_k + |a_k| \geq 0$

Since

$$0 \leq a_k + |a_k| \leq 2 \cdot |a_k|$$

and $\quad \sum_{k=1}^{\infty} 2|a_k|$ Converges,

by Comparison test, we have

$\sum_{k=1}^{\infty} (a_k + |a_k|)$ Converges.

$\Rightarrow$

$$\underbrace{\sum_{k=1}^{\infty} (a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|}_{= \sum_{k=1}^{\infty} a_k} \text{ Converges}$$

#

Example

① $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \cdots$$

Since

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges}$$

we know $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ is absolutely convergent

$\Rightarrow$ it converges. #

② $$\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$$

is absolutely convergent

$\Rightarrow$ Convergent.

③ $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is NOT absolutely convergent since

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.}$$

Thm (Thm 12.5.3, Alternating series)

Let $(a_k)_{k=1}^{\infty}$ be a decreasing seq. of positive numbers.

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{Converges}$$

$$\Longleftrightarrow \quad \lim_{k\to\infty} a_k = 0$$

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is conditionally convergent

idea of pf

Let $S_n = \displaystyle\sum_{k=1}^{n} (-1)^{k+1} a_k$

Consider

$$(S_{2m-1})_{m=1}^{\infty} = S_1, S_3, S_5, \ldots$$

and

$$(S_{2m})_{m=1}^{\infty} = S_2, S_4, S_6, \ldots$$

NOTE:

$$S_{2m-1} = a_1 + \overset{\le 0}{(-a_2 + a_3)} + \overset{\le 0}{(a_4 + a_5)} + \cdots + \overset{\le 0}{(-a_{2m-2} + a\ )}$$

$+$

$$S_{2m+1} = a_1(-a_2 +) \dots + a_{2m-3}(-a_{2m-2} + a_{2m+1})$$

$$\boxed{- a_{2m} + a_{2m+1}} \leq 0$$

So $(S_{2m-1})_{m=1}^{\infty}$ is decreasing.

Since $\lim\limits_{k\to\infty} a_k = 0$, $(a_k)_{k=1}^{\infty}$ is bounded

$\Rightarrow \exists M$ s.t. $\boxed{a_k \geq M}$ $\forall k$

$$\Rightarrow S_{2m-1} = \overset{\geq 0}{(a_1 - a_2)} + \overset{\geq 0}{(a_3 - a_4)} + \dots$$

$$+ \overset{\geq 0}{(\quad)} + \underline{\underline{a_{2m-1}}} \geq M$$

$$\geq M$$

So $(S_{2m-1})_{m=1}^{\infty}$ is decreasing.

bounded below $\Rightarrow$

$\lim\limits_{m\to\infty} S_{2m-1}$ exists. $\overset{\text{let}}{=} \underline{\underline{L}}$

$\Rightarrow$ $\quad\quad\quad\quad L \quad\quad 0$

$$\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} \left(\boxed{S_{2m-1}} - \underline{a_{2m}}\right)$$

$$= L$$

"by ε argument"

$$\boxed{\Rightarrow} \quad \lim_{n\to\infty} S_n = L \qquad \#$$

<u>Remark</u>

A rearrangement of $\sum_{k=1}^{\infty} a_k$ is a series that has exactly same terms but in a different order. For example

$$a_1 + a_3 + a_5 + a_2 + a_{10} + a_4 + a_6 + \cdots.$$

① All rearrangements of an absolutely convergent series converge to the same limit.

② If a series is conditionally convergent then it can be arranged to converge to ANY number, or to diverge to $-\infty$ or $+\infty$. or to oscillate between any two numbers.

§ Taylor expansion

Goal: "approach" complicated functions by polynomials.

Q: How can we find a seq. of polynomials that converges to a given function $f(x)$?

Idea: Try to find polynomials which has same derivatives as $f(x)$ up to n-th order.

$P(x)$

What:

Want.

$$f(x) \approx (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$$

Assume

$$f(0) = p(0) = a_0$$

$$f'(0) = p'(0) = \left(\overset{\overset{a_1}{\parallel}}{a_1} + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}\right)\Big|_{x=0}$$

$$f''(0) = p''(0) = 2a_2$$

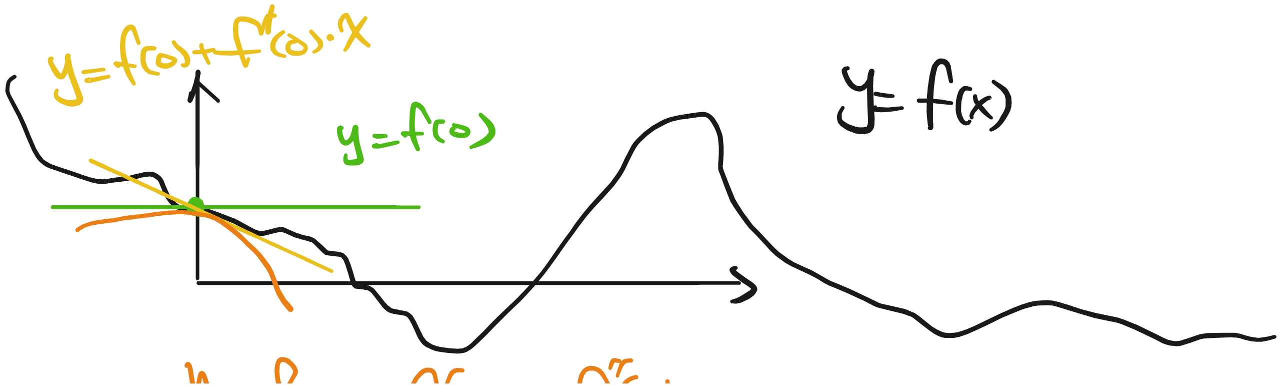
$$\vdots \qquad \vdots$$

$$f^{(n)}(0) = p^{(n)}(0) = n! \cdot a_n$$

Then

$$\boxed{p(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.}$$

" best deg-n polynomial approximation of f(x) around $x = 0$ "



$$y = f(0) + f''(0)x + \frac{f'(0)}{2}x^2$$

If we consider a larger n, then we get a "better" approximation.

As $n \to \infty$, we have

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

$$\parallel \leftarrow \text{hope}$$

$$f(x)$$

Q:

(i) For what x, does the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ Converge ?

(ii) If $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converges, does it converge to $f(x)$?

Thm (Taylor's Thm, Thm 12.6.1)

Assume $I = (a,b)$, $0 \in I$,

If f has $n+1$ continuous derivatives on I, then, $\forall x \in I$,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

Pf

<u>$n=0$</u>:

$$f(x) = f(0) + R_0(x)$$

$$\Rightarrow \quad R_0(x) = f(x) - f(0)$$

$$R_0(x) = \frac{1}{0!}\int_0^x f'(t) \cdot dt$$

Recall

$$\int u \cdot v' \, dt = uv - \int v \cdot u' \, dt$$

So

$$f(x) = f(0) + \int_0^x \underline{f'(t)} \, dt.$$

<u>$n=1$</u>:

Let $u(t) = f'(t)$ $\qquad u'(t) = f''(t)$

$v(t) = t - x$ $\qquad v'(t) = 1$

$$f(x) = f(0) + \boxed{f'(t)\cdot(t-x)\Big|_{t=0}^{x}} = f'(0)\cdot x$$

$$- \int_0^x (t-x)\cdot f''(t)\,dt$$

$$R_1(x)$$

$$= f(0) + f'(0)\cdot x + \underline{\overset{\|}{\int_0^x f''(t)\cdot(x-t)'\,dt}}$$

Assume the thm is true for $k = 0, 1, 2, \cdots, n-1$. Then

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$+ \boxed{\frac{1}{(n-1)!}\int_0^x \underbrace{f^{(n)}(t)}_{u(t)} \cdot \underbrace{(x-t)^{n-1}}_{v'(t)}\,dt} \quad R_{n-1}(x)$$

$$\|$$

$$\frac{1}{(n-1)!}\left(f^{(n)}(t)\cdot \frac{(x-t)^n}{n}\cdot(-1)\Big|_0^x + \int_0^x f^{(n+1)}(t)\,\frac{(x-t)^n}{+n}\,dt \right)$$

$$= \frac{1}{n!}f^{(n)}(0)\cdot x^n + \underline{\frac{1}{n!}\int_0^x f^{(n+1)}(t)\cdot(x-t)^n\,dt}$$

$\Rightarrow$ Thm is also true for $\overset{R_n(x)}{k=n}$ #

Another expression of $R_n(x)$

Thm (Second mean value thm, Thm 5.9.3)

If u and v are continuous on $[a,b]$ and $v \geq 0$ on $[a,b]$, then $\exists\ c \in [a,b]$ s.t.

$\boxed{m} \cdot \left(\sum_i v(i)\right)$

$v(1)+v(2)+v(3)=5$

$$\sum_i u(i)\cdot v(i) \longleftrightarrow \int_a^b u(x)\cdot v(x)\,dx = \overset{m}{\underline{u(c)}}\cdot \int_a^b v(x)\,dx$$

e.g. $i = 1, 2, 3$

$v(1)=2$, $v(2)=2$, $v(3)=1$

score: $u(1)$: science $u(2)$: math $u(3)$: English

Such $u(c)$ is called the **v-weight average** of u on $[a,b]$

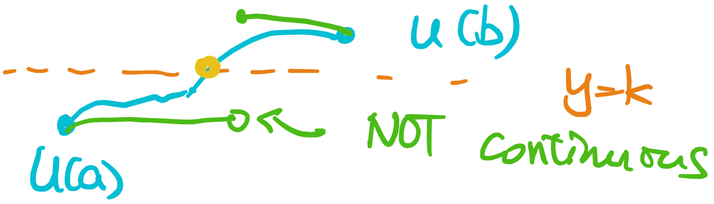
pf

If $v \equiv 0$, then "=" is true.

Assume $v \not\equiv 0 \Rightarrow \int_a^b v(x)\,dx > 0$

Recall (IVT) on $[a,b]$

If u is continuous, k is between $u(a)$ and $u(b)$, then $\exists\ c \in [a,b]$ s.t. $u(c)=k$



By the extreme value thm, u takes a minimum value $\bar{m} = u(C_0)$ and a maximum value $\bar{M}$ on $[a,b]$.
$= u(C_1)$

$$\bar{m} \cdot v(x) \leq u(x) \cdot \overset{\geq 0}{v(x)} \leq \bar{M} \cdot v(x)$$

$$\forall x \in [a,b]$$

$$\Rightarrow \bar{m} \cdot \underbrace{\int_a^b v(x)\,dx}_{>0} \leq \int_a^b u(x) \cdot v(x)\,dx \leq \bar{M} \cdot \underbrace{\int_a^b v(x)\,dx}_{>0}$$

$$\Rightarrow u(C_0) = \bar{m} \leq \overset{=k}{\boxed{\frac{\int_a^b u(x) \cdot v(x)\,dx}{\int_a^b v(x)\,dx}}} \leq \bar{M} = u(C_1)$$

By the intermediate value thm, $\exists\ c$ between C_0 and C_1 s.t.

$$\int_a^b u(x) \cdot v(x)\,dx$$

$$u(c) = \frac{\int_a}{\int_a^b v(x)\,dx} \qquad \#$$

Apply the thm to

$$R_n(x) = \frac{1}{n!}\int_{\underline{0}}^{x} f^{(n+1)}(t)\cdot(x-t)^n\,dt$$

with

$$u(t) = \begin{cases} f^{(n+1)}(t) & \text{if } x \geq 0 \\ (-1)^n\cdot f^{(n+1)}(t) & \text{if } x < 0 \end{cases}$$

$t \in [0, x]$ if $x \geq 0$

$t \in [x, 0]$ if $x < 0$

$$V(t) = \begin{cases} \frac{1}{n!}(x-t)^n & \text{if } x \geq 0. \\ \frac{1}{n!}(t-x)^n & \text{if } x < 0 \end{cases} \quad \geq 0$$

Then we get

(i) if $x \geq 0$, $\exists c$ s.t.

$$-\frac{(x-t)^{n+1}}{(n+1)!}\Bigg|_{t=0}^{x}$$

$\|$

$$R_n(x) = f^{(n+1)}(c)\cdot\underline{\int_0^x \frac{1}{n!}(x-t)^n\,dt}$$

$$= f^{(n+1)}(c)\cdot\frac{x^{n+1}}{(n+1)!}$$

$$\frac{(t-x)^{n+1}}{}\Bigg|^{x}$$

(ii) if $x<0$, $\exists\, c$ s.t

$(n+1)!$ $\quad |t=0$

$$R_n(x) = (-1)^n f^{(n+1)}(c) \cdot \underline{\int_0^x \frac{(t-x)^n}{n!}\, dt} \overset{\shortparallel}{-} \frac{(-x)^{n+1}}{(n+1)!}$$

$$= \cancel{(-1)^n} f^{(n+1)}(c) \left(\cancel{-} \frac{(\cancel{-}x)^{n+1}}{(n+1)!} \right)$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

Conclusion:

Thm

The reminder $R_n(x)$ can written as

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n\, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some $\underset{\sim}{c} \overset{= c(x)}{}$ between 0 and x.

In other words,

$$f(x) = f(0) + \frac{f'(0)}{\ } x + \frac{f''(0)}{\ } x^2 \quad \frac{f'''(0)}{\ } x^3$$

$$f(x) = f(0) + \frac{\quad}{1!} x + \frac{\quad}{2!} x + \frac{\quad}{3!} x$$

$$+ \cdots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

with $c = c(x)$ between 0 and x.

Remark

The series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots$$

Converges to $f(x)$ if

$$\lim_{n \to \infty} R_n(x) = 0$$

Thm

For any real number x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

n

pf

By the previous thm,

$$e^x = \underline{\overset{1}{e^0}} + \frac{\boxed{(e^x)'\big|_{x=0}}^{=e^0=1}}{1!}x^1 + \frac{\boxed{(e^x)''\big|_{x=0}}^{=e^0=1}}{2!}x^2 + \cdots$$

$$+ \frac{\boxed{(e^x)^{(n)}\big|_{x=0}}^{=1}}{n!}x^n + R_n(x)$$

$$R_n(x) = e^x - \left(\sum_{k=0}^{n} \frac{x^k}{k!}\right)$$

$$= \frac{\overset{e^x =}{\boxed{(e^x)^{(n+1)}}}\big|_{x=c}}{(n+1)!} \cdot x^{n+1}$$

$$= \frac{e^c}{(n+1)!} \cdot x^{n+1}$$

for some c between 0 and x

Note that

$$|R_n(x)| = \left|\frac{e^c}{(n+1)!} x^{n+1}\right| \leftarrow \quad |e^c| \le \begin{cases} e^x & \text{if } x \ge 0 \\ e^0 & \text{if } x < 0 \end{cases} \le e^{|x|}$$

$$\le \boxed{\frac{|x|^{n+1}}{(n+1)!}} \xrightarrow{} 0 \cdot e^{|x|}$$

$\forall x$

$$\longrightarrow 0 \quad \text{as} \quad n \to \infty$$

Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \#$$

<u>Thm</u>

$\forall x \in \mathbb{R}$,

(i) $$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(ii) $$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

pf

$\sin(0) = 0$, $(\sin x)'\big|_{x=0} = \cos 0 = 1$

$(\sin x)''\big|_{x=0} = -\sin 0 = 0$, $(\sin x)'''\big|_{x=0} = -\cos 0 = -1$

${}^{(4)}\big|$

$(\sin x)' |_{x=0} = \sin 0$, ...

$\Rightarrow$ for $f(x) = \sin x$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

many possibilities

$$R_n(x) = \sin x - \left(0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \cdots + \quad x^n\right)$$

sin c or cos c or -sin c or -cos c $\longrightarrow$

$$= \frac{\boxed{(\sin x)^{(n+1)}|_{x=c}}}{(n+1)!} x^{n+1}$$ for some c between 0 and x.

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{\forall x} 0 \quad \text{as } n \to \infty$$

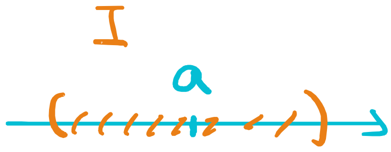
$$\Rightarrow \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

\#

Instead of $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, we also can consider

$$\sum^{\infty} f^{(k)}$$

$$\sum_{k=0} \frac{f\ (a)}{k!} (x-a)$$



Thm (Thm12.7.1)

If f has $n+1$ continuous derivatives on an open interval I that contains a, then for each $x \in I$,

$$f(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1}$$

for some $\underset{\sim}{c}$ between a and x

$\shortparallel$

$C(x)$ depending on x.

Example

Consider the polynomial

$$f(x) = 4x^3 - 3x^2 + 5x - 1$$

Write it in the form

$$f(x) = a_3 \cdot (x-2)^3 + a_2 \cdot (x-2)^2 + a_1 (x-1) + a_0$$

Sol

$$f(2) = 29 \qquad f^{(n)}(2) = 0 \quad \forall n \geq 4$$

$$f'(2) = 41 \qquad \Downarrow$$

$$f''(2) = 42 \qquad R_n(x) = 0 \quad \forall n \geq 3$$

$$f'''(2) = 24$$

So

$$f(x) = 29 + 41 \cdot (x-2) + \frac{42}{2!} \cdot (x-2)^2 + \frac{24}{3!} \cdot (x-2)^3$$

$\#$