Calculus, week 3, Spring 2025

Recall

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$$= \begin{cases} \dfrac{1}{p-1}, & p>1 \\ \text{diverges} & p\le 1 \end{cases}$$

Def

Integrals of functions that become infinite at a point in the interval of integration are called improper integrals (of type II)

(i) If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

(ii) If $f(x)$ is continuous on $[a, b)$

(ii) If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

(iii) If $f(x)$ is discontinuous at $c \in (a,b)$ and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

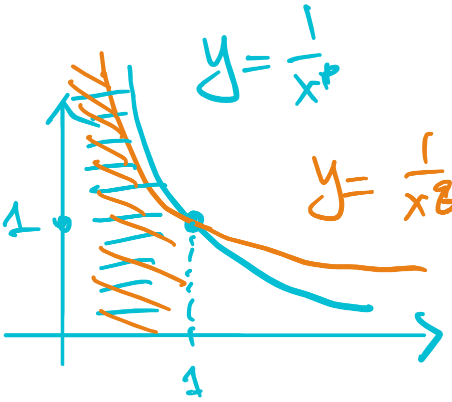
In each case, if the limit exists (and is finite), we say the improper integral converges and the limit is the value of the improper integral. If the limit does not exist, we say the improper integral diverges.

Thm

$$\int^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}, & p<1 \end{cases}$$

$\int_0 \quad x$

diverges, $\quad p \geq 1$



$\left(\frac{x^{-p+1}}{-p+1}\right)' =$

Pf

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^p}\,dx$$

$$= \lim_{c\to 0^+} \left(\frac{x^{-p+1}}{-p+1} \Big|_c^1 \right)$$

assume $p \neq 1$

$$= \lim_{c\to 0^+} \left(\frac{1}{1-p} - \frac{c^{-p+1}}{1-p} \right)$$

converges to 0 $\quad -p+1>0$

diverges $\quad -p+1<0$

$$= \begin{cases} \frac{1}{1-p}, & p<1 \\ \text{diverges}, & p>0 \end{cases}$$

If $p=1$,

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx$$

$$= \lim_{c\to 0^+} \left(\ln x \Big|_c^1 \right)$$

$$= \lim_{c\to 0^+} \left(0 - \ln c \right)$$

diverges. #

<u>Example</u>

① $$\int_0^1 \frac{1}{1-x}\,dx = \lim_{c\to 1^-} \int_0^c \frac{1}{1-x}\,dx$$

$$= \lim_{c\to 1^-} \left(-\ln|1-x| \Big|_0^c \right)$$

$$= \lim \left(-\ln|\underset{0}{1-c}| \right)$$ diverges

$= \lim_{c \to 1^-} \left(\ln \ldots \right) = $ diverges. #

discontinuous at $x = 1$

② $$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx \qquad \left(3(x-1)^{\frac{1}{3}}\right)' = \frac{1}{(x-1)^{\frac{2}{3}}}$$

$$= \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$= \lim_{c \to 1^-} \int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \lim_{d \to 1^+} \int_d^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

$$= \lim_{c \to 1^-} \underbrace{\left(3(x-1)^{\frac{1}{3}} \Big|_0^c \right)}_{3\sqrt[3]{c-1} \,+\, 3\cancel{\sqrt[3]{-1}}\,\to +1} + \lim_{d \to 1^+} \underbrace{\left(3(x-1)^{\frac{1}{3}} \Big|_d^3 \right)}_{3\sqrt[3]{2} \,-\, 3\sqrt[3]{d-1}}$$

$3\sqrt[3]{c-1} \to 0$ $\qquad$ $3\sqrt[3]{d-1} \to 0$

$$= 3 + 3\sqrt[3]{2} \quad \#$$

Q: Which improper integrals converge?

Thm (11.7.2)

Let f and g be continuous function on $[a, \infty)$.

If

$$0 \le f(x) \le g(x) \quad \forall\, x \ge a$$

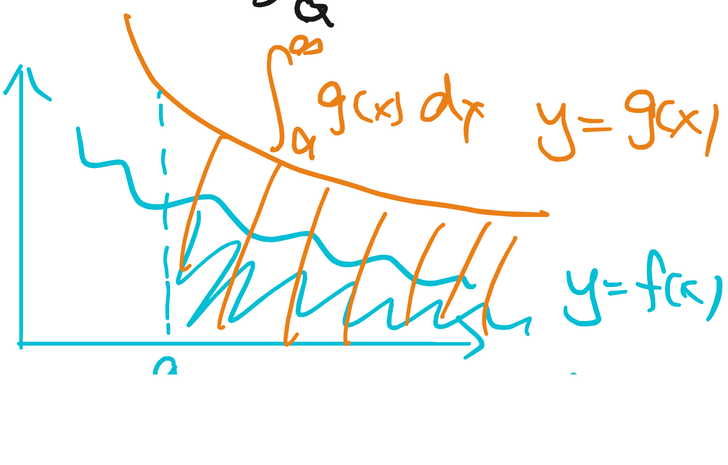
then

(i) $\displaystyle\int_a^\infty g(x)\,dx$ converges

$$\Rightarrow \int_a^\infty f(x)\,dx \text{ converges}$$

(ii) $\displaystyle\int_a^\infty f(x)\,dx$ diverges

$$\Rightarrow \int_a^\infty g(x)\,dx \text{ diverges}$$



$\int_a^\infty f(x)\,dx$

<u>Thm</u>

Let f, g be positive, continuous on $[a, \infty)$

If

$\left(\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \right)$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty.$$

then

$\left(\int_a^\infty g(x)\,dx \text{ converges} \Rightarrow \int_a^\infty f(x)\,dx \text{ converges} \right)$

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverges.

<u>Example</u>

Determine the following integrals converge or diverge.

① $\int^\infty \frac{\sin^2 x}{}\,dx$ converges because

$\int_1 \frac{\quad}{x^2}\, dx$ converges because

(a) $$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \forall x \geq 1$$

and

(b) $$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges}$$ #

(2) $$\int_1^\infty \frac{1}{\sqrt{x^2 - \frac{1}{2}}}\,dx \text{ diverges because}$$

(a) $$\frac{1}{\sqrt{x^2 - \frac{1}{2}}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x} \quad \forall x \geq 1$$

and

(b) $$\int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$ #

(3) $$\int_1^\infty \frac{1}{1+2x^2+x}\,dx \text{ converges because}$$

(a) $$\lim_{x\to\infty} \frac{\frac{1}{1+x+2x^2}}{\frac{1}{x^2}} = \frac{1}{2} \neq 0$$

and

(b) $$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges}$$ #

④ $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because

(a)
$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{\cancel{x}}}{\frac{1}{\cancel{x}}} = 1 \neq 0$$

$e^{-x} \to 0$

and

(b) $\int_1^\infty \frac{1}{x}\,dx$ diverges #

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§ Infinite series

In this chapter, we will consider

" $a_0 + a_1 + a_2 + a_3 + \cdots\cdots$ "

Let $(a_k)_{k=0}^{\infty}$ be a seq.

Consider the seq.

$$\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$$

of partial sums:

$$a_0,\ (a_0+a_1),\ (a_0+a_1+a_2),\ \cdots$$

Def

We say that the series 級數 $\sum_{k=0}^{\infty} a_k$ converges to L if the seq.

$$\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$$

converges to L.

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = L$$

The number L is called the sum of the series.

We say the series $\sum_{k=0}^{\infty} a_k$ diverges if the seq. $\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ diverges.

Example

① $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \, ?$

Sol

$$= \frac{1}{k+1} - \frac{1}{k+2}$$

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$

$$= \left(\frac{1}{0+1} - \cancel{\frac{1}{0+2}} \right) + \left(\cancel{\frac{1}{1+1}} - \cancel{\frac{1}{1+2}} \right) + \left(\cancel{\frac{1}{2+1}} - \cancel{\frac{1}{2+2}} \right)$$

$$\cancel{+} \cdots \cancel{+} \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2} \right)$$

$$= 1 - \frac{1}{n+2} \longrightarrow 1 \quad \text{as} \quad n \to \infty$$

So $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1$ #

② $\displaystyle\sum_{k=0}^{\infty} (-1)^k = \, ?$

Sol

$$\sum_{k=0}^{n} (-1)^k = \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cdots + \underline{(-1)^n}$$

$$= \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

if n is odd

So
$$\sum_{k=0}^{\infty} (-1)^k \text{ diverges} \quad \#$$

③ $\displaystyle\sum_{k=0}^{\infty} x^k = ?$ 等比級數 (geometric series) 幾何級數

<u>sol</u>

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

$$= \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1 \end{cases}$$

So

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

<u>Remark</u>

Let p be a positive integer.

$$\sum_{k=0}^{\infty} a_k \quad \text{converges}$$

$$\Leftrightarrow \sum_{k=p}^{\infty} a_k \quad \text{converges}$$

If they converge,

$$\sum_{k=0}^{\infty} a_k = \sum_{k=p}^{\infty} a_k + \sum_{k=0}^{p-1} a_k$$

Thm (Thm 12.2.4, Thm 12.2.5)

Let $a_k, b_k, \alpha, \beta \in \mathbb{R}$.

(i) If $\sum_{k=0}^{\infty} a_k$ converges to L

$\sum_{k=0}^{\infty} b_k$ " M,

then

$$\sum_{k=0}^{\infty} (\alpha \cdot a_k + \beta \cdot b_k) \quad \text{converges}$$

to $\alpha \cdot L + \beta \cdot M$

NOTE: $\left(\sum_{k=0}^{n} a_k\right)\cdot\left(\sum_{k=0}^{n} b_k\right) \neq \sum_{k=0}^{n} a_k b_k$ ~~L.M~~

L.M

e.g. $(a_0+a_1)(b_0+b_1) = a_0b_0 + a_0b_1 + a_1b_0 + a_1b_1$

$\neq a_0b_0 + a_1b_1$

(ii) If $\sum_{k=0}^{\infty} a_k$ Converges, then

$$\lim_{k\to\infty} a_k = 0.$$

Equivalently, $\lim_{k\to\infty} a_k \neq 0$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ diverges.}$$

pf

$$\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} a_k = 0$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n+1}$

$$= \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right) - \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} a_k\right)$$

$$= \lim\left(\sum^{n} a \quad \sum^{n-1} a \quad\right)$$

$$-\lim_{n\to\infty}\left(\sum_{k=0} u_k - \sum_{k=0} u_k\right)$$

$$= \lim_{n\to\infty} a_n = 0 \qquad \#$$

Example

Recall
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x|<1$$

① $$\sum_{k=0}^{\infty}\left(\left(\tfrac{1}{2}\right)^k + 2\cdot\left(\tfrac{1}{3}\right)^k\right)$$

$$= \sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k + 2\cdot\sum_{k=0}^{\infty}\left(\tfrac{1}{3}\right)^k$$

$$= \frac{1}{1-\frac{1}{2}} + 2\cdot\frac{1}{1-\frac{1}{3}} = 5 \qquad \#$$

② $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges because

$$\lim_{k\to\infty} \frac{\cancel{k}\ 1}{1\ \cancel{k}+\cancel{1}\,\frac{1}{k}} = 1 \neq 0 \qquad \#$$

$$\frac{1}{k} \to 0$$

NOTE: $\displaystyle\lim_{k\to\infty}\frac{1}{\sqrt{k}} = 0$

③ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges because

$$\sum^{n} \frac{1}{\ldots} = \frac{1}{\ldots} + \frac{1}{\ldots} + \cdots + \frac{1}{\ldots}$$

$\overline{k=1} \quad \sqrt{k} \qquad \sqrt{1} \quad \sqrt{2} \qquad\qquad \sqrt{n}$

$$\geq \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ times}}$$

$$= n \cdot \frac{1}{\sqrt{n}} = \sqrt{n} \quad \to +\infty$$

$$\Rightarrow \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)_{n=1}^{\infty} \text{ is unbounded} \qquad \#$$

§ Convergence of series

Thm (Thm 12.3.1)

A series $\sum_{k=0}^{\infty} a_k$ with nonnegative terms converges if and only if the seq. $\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ is bounded above.

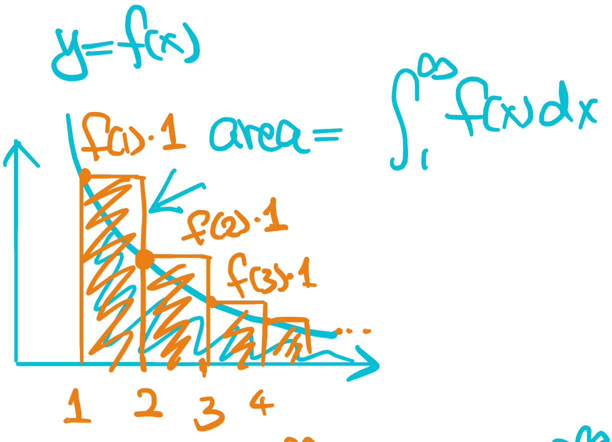
Thm (Integral test, Thm 12.3.2)

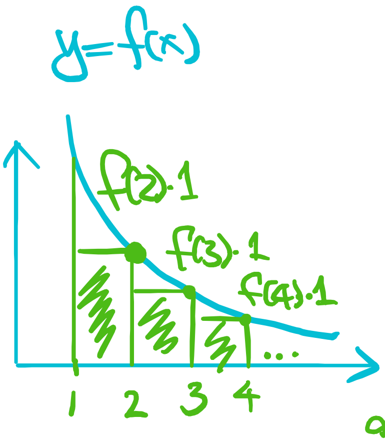
If $f = f(x)$ is continuous, positive and decreasing on $[1, \infty)$, then

$$\sum^{\infty} f(k) \quad \text{converges}$$

$\sum_{k=1} \underbrace{f(k)}_{f(k)\cdot 1}$ converges

$\Longleftrightarrow \int_1^{\infty} f(x)\,dx$ Converges





$$\text{area} = \sum_{k=1}^{\infty} f(k) \geq \int_1^{\infty} f(x)dx \geq \text{area} = \sum_{k=2}^{\infty} f(k)$$

$$\sum_{k=2}^{n} f(k) \leq \int_1^{n} f(x)\,dx \leq \sum_{k=1}^{n-1} f(k)$$

Example (p-series)

① Consider $f(x) = \frac{1}{x}$ which is positive, continuous, decreasing on $[1, \infty)$.

Since $\int_1^{\infty} \frac{1}{x}\,dx$ $= \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty}(\ln b - \cancel{\ln 1})$ diverges, diverges,

by the integral test, we know

$$\sum_{k=1} \frac{1}{k} \text{ diverges.} \qquad \#$$

② Consider $f(x) = \frac{1}{x^p}$ which is continuous, positive, decreasing ($p>0$). on $[1, \infty)$.

Since

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges} \Leftrightarrow p>1$$

we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow p>1.$$

<u>Thm</u> (Basic Comparison test, Thm 12.3.6)

Let a_k and b_k be nonnegative numbers. Suppose

$$a_k \leq b_k \quad \text{for } k \text{ sufficient large.}$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then

$$\sum_{k=1}^{\infty} a_k \text{ converges}$$

(ii) If $\sum_{1}^{\infty} a_k$ diverges, then

$$\sum_{k=1}^{\infty} b_k \text{ diverges.}$$

Example

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^3+1}$ Converges because

$$\frac{1}{2k^3+1} = \frac{1}{k^3+(k^3+1)} < \frac{1}{k^3} \quad \forall k \geq 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \text{ Converges.} \qquad \#$$

② $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^5+5k^4+7}$ Converges because

$$\frac{k^3}{k^5+5k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2} \qquad \forall k \geq 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ Converges} \qquad \#$$

③ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3k+1}$ diverges because

$$\frac{1}{3k+1} = \frac{1}{4k-\underbrace{(k-1)}_{\geq 0}} \geq \frac{1}{4k} \quad \forall\, k \geq 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{4k} = \frac{1}{4} \cdot \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.} \qquad \#$$

Thm (Limit comparison test, Thm 12.3.7)

Let a_k, b_k be positive numbers. If

$$\lim_{k\to\infty} \frac{a_k}{b_k} \neq 0$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\Longleftrightarrow \sum_{k=1}^{\infty} b_k \text{ converges.}$$

Example

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k-3}$ converges because

$$\lim \frac{\frac{1}{5^k-3}}{\cdots} = \lim \frac{1}{1 - \frac{3}{\cdots}} = 1 \neq 0$$

$\lim_{k\to\infty} \frac{1}{5^k}$ $\qquad$ $k\to\infty$ $\qquad 1 - \left(5^k\right) \to 0$

and $\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges #

② $\sum_{k=1}^{\infty} \frac{3k^2+2k+1}{k^3+1}$ diverges because

$$\lim_{k\to\infty} \frac{\frac{3k^2+2k+1}{k^3+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\overbrace{(3k^3+2k^2+k)\frac{1}{k^3}}^{3+\frac{2}{k}+\frac{1}{k^2}}}{\underbrace{(k^3+1)\frac{1}{k^3}}_{1+\frac{1}{k^3}}} = 3 \neq 0$$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges #

③ $\sum_{k=1}^{\infty} \frac{2k+5}{\sqrt{k^6+3k^3}}$ Converges because

$$\lim_{k\to\infty} \frac{\frac{2k+5}{\sqrt{k^6+3k^3}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\overbrace{(2k^3+5k^2)\frac{1}{k^3}}^{2+\frac{5}{k}}}{\underbrace{\left(\sqrt{k^6+3k^3}\right)\cdot\frac{1}{k^3}}_{\sqrt{1+\frac{3}{k^3}}}} = 2 \neq 0$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges. #

Thm (Root test, Thm 12.4.1)

Let a_k be nonnegative numbers.

$\sum_{k=1}^{\infty} x^k$

Suppose

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \rho.$$

(i) If $\rho < 1$, then $\sum_{k=1}^{\infty} a_k$ Converges.

(ii) If $\rho > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(iii) If $\rho = 1$, then $\sum_{k=1}^{\infty} a_k$ may converge or diverge.

e.g.

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\lim_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = \frac{1}{1} = 1$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges, $\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^2}} = \frac{1}{\lim_{k\to\infty} (\sqrt[k]{k})^2} = 1$

pf

If $\rho < 1$, choose μ, $\rho < \mu < 1$

Since $\lim_{k\to\infty} (a_k)^{\frac{1}{k}} = \rho < \mu$,

for $\varepsilon = \mu - \rho > 0$. $\exists \tilde{K}$ s.t.

$$|(a_k)^{\frac{1}{k}} - \rho| < \mu - \rho \quad \forall\, k \geq \tilde{K}$$

$$\rho-(\mu-\rho) < (a_k)^{\frac{1}{k}} < \rho+(\mu-\rho) = \mu < 1$$

$\Rightarrow$

$$0 \leq a_k < \mu^k \qquad \forall\ k \geq \tilde{K}$$

Since $\sum_{k=1}^{\infty} \mu^k$ $(\mu<1)$ converges, by the comparison test,

$$\sum_{k=1}^{\infty} a_k$$

also converges. $\Rightarrow$ (i) #

<u>Example</u>

① $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$ converges because

$$\lim_{k\to\infty} \left(\frac{1}{(\ln k)^k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{\ln k} = 0 < 1$$

#

② $\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ diverges because

$$\lim_{k\to\infty}\left(\frac{\;}{k^3}\right) = \lim_{k\to\infty}\frac{\;}{(\sqrt[k]{k})^3} = 2 > 1 \quad \#$$

<u>Thm</u> (Ratio test, Thm 12.4.2)

Let a_k be positive numbers. Suppose that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lambda.$$

(i) If $\lambda < 1$, then $\sum_{k=1}^{\infty} a_k$ converges

(ii) If $\lambda > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(iii) If $\lambda = 1$, then $\sum_{k=1}^{\infty} a_k$ may converge or diverge.

$\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. $\lambda = \lim_{k\to\infty}\frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k}{k+1} = 1$

$\sum_{k=1}^{\infty}\frac{1}{k^2}$ Converges $\lambda = \lim_{k\to\infty}\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 = 1$

<u>Example</u>

① $\sum_{k=1}^{\infty}\frac{1}{k!}$ Converges because

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k\to\infty}\frac{1}{k+1} = 0 < 1$$

$k!$ #

② $\sum_{k=1}^{\infty} \frac{k}{10^k}$ Converges because

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{k+1}{10^{k+1}}}{\frac{k}{10^k}} = \lim_{k\to\infty} \frac{\frac{k+1}{k}}{10} = \frac{1}{10} < 1$$

#

③ $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges because

$$\lambda = \lim_{k\to\infty} \frac{\frac{(k+1)^{k+1}}{\cancel{(k+1)!}\ k+1}}{\frac{k^k}{\cancel{k!}}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$$

$$= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$ #

<u>Remark</u>

① $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$

$\lambda = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ might not Converge.

ρ exists $\not\Rightarrow$ λ exists.

② So far we assume $a_k \geq 0$ in

$$\sum_{k=1}^{\infty} a_k$$

<u>Next</u>: what if $a_k < 0$?