Calculus, week 2, Spring 2025

Recall (Pinching Thm)

Suppose for all n sufficiently large,

$$a_n \leq b_n \leq c_n.$$

If $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then

$$\lim_{n\to\infty} b_n = L.$$

Example

$$\lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2} = ?$$

Sol

NOTE:

$$(2+\tfrac{1}{n})^2$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$2 = \sqrt{4} \leq \sqrt{4+\left(\frac{1}{n}\right)^2} \leq \sqrt{\boxed{4+2\cdot 2\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2}}$$

$$\begin{array}{c} \sqrt{4+2\cdot 2\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2} \\ \parallel \\ 2+\frac{1}{n} \end{array}$$

Since

$$\lim 2 = 2 = \lim \left(2+\frac{1}{n}\right),$$

$n\to\infty$ $n\to\infty$

we have

$$\lim_{n\to\infty}\sqrt{4+\left(\frac{1}{n}\right)^2} = 2 \qquad \#$$

This limit $\lim_{n\to\infty}\sqrt{4+\left(\frac{1}{n}\right)^2}$ also can be computed by the following thm:

<u>Thm</u> (Thm 11.3.12) ($\lim_{n\to n} C_n = C$, $C_n \geq C$)

Suppose $\lim_{n\to\infty} C_n = C$. If f is (right continuous) continuous at c, then

$$\lim_{n\to\infty} f(C_n) = f\left(\lim_{n\to\infty} C_n\right) = f(C).$$

<u>Example</u>

① $\lim_{n\to\infty}\sqrt{4+\left(\frac{1}{n}\right)^2} = ?$

<u>Sol2</u>

Consider

$$C_n = 4+\left(\frac{1}{n}\right)^2 \longrightarrow \overset{c}{\overset{\shortparallel}{4}} \quad \text{as } n\to\infty$$

$f(x) = \sqrt{x}$ is cont at 4

$f(x) = \sqrt{x}$ is continuous at 4.

So, by Thm,

$$\lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2}$$

$$= f\left(\lim_{n\to\infty} c_n\right) = f(4) = \sqrt{4} = 2 \ \#$$

②

- $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\overset{=0}{\cancel{\lim_{n\to\infty} \frac{\pi}{n}}}\right)$

  $= \sin 0 = 0$

- $\lim_{n\to\infty} e^{\frac{\pi}{n}} = e^{\lim_{n\to\infty} \frac{\pi}{n}} = e^{0} = 1$

- $\lim_{n\to\infty} \tan^{-1}\left(\frac{\pi}{n}\right) = \tan^{-1}\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \tan^{-1} 0 = 0$

Recall

Polynomials, $\sin x$, $\cos x$, e^{x}, $\tan^{-1} x$ are continuous on $(-\infty, \infty)$.

③

$\lim_{n\to\infty} \frac{2-\cancel{\left(\frac{1}{n}\right)}^{\to 0}}{1} = 2$

- $\lim_{n\to\infty} \sqrt{\frac{2n-1}{n}} = \sqrt{\boxed{\lim_{n\to\infty} \frac{2n-1}{n}}}$

  $= \sqrt{2}$

Recall

$\sqrt{x}$, $\ln x$ are continuous on $(0, \infty)$

$|x|$ is continuous on $(-\infty, \infty)$

- $\lim \ln\left(\frac{2n-1}{n}\right) = \ln 2$

- $\lim_{n\to\infty} \left| \frac{2n-1}{2} \right| = |2| = 2$ #

Recall that we've learned

$$\lim_{x\to c} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x).$$

They are related to $\lim_{n\to\infty} a_n$ in the following way:

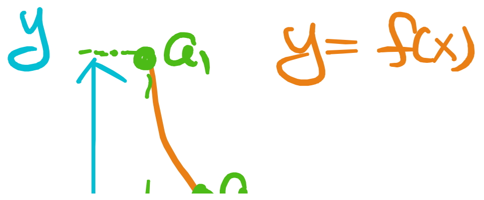
<u>Thm</u>

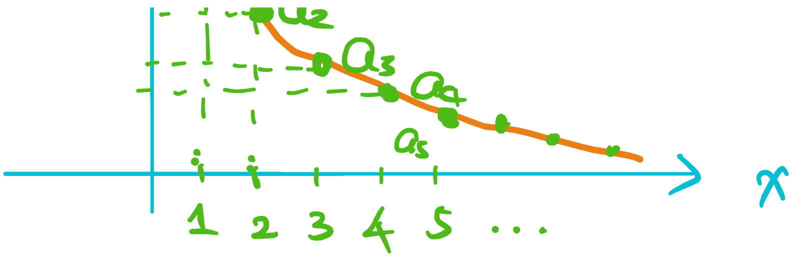
Let f be a function, $c \in \mathbb{R}$.

(i) If $\lim_{x\to\infty} f(x) = L$, ← $x \in \mathbb{R}$

then, for $a_n = f(n)$, ↙ n are positive integers

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$$



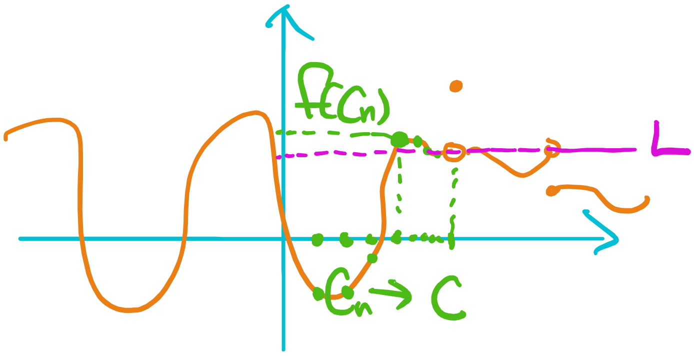


(ii) If $\lim_{x \to c} f(x) = L$ and if $(c_n)_{n=1}^{\infty}$ is any seq. which converges to c

$$\lim_{n \to \infty} c_n = c$$

then

$$\lim_{n \to \infty} f(c_n) = L = \lim_{x \to c} f(x)$$



Exercise

Prove that

$$\lim \sin\left(\frac{1}{x}\right)$$

$x\to 0$

does NOT exist.

pf

Assume $\lim_{x\to 0} \sin\frac{1}{x}$ exists, $= L$.

Consider

$$\frac{1}{n\pi} \longrightarrow 0$$

$$\frac{1}{2n\pi+\frac{\pi}{2}} \longrightarrow 0$$

Since

$$\sin\left(\frac{1}{\frac{1}{n\pi}}\right) = \sin(n\pi) = 0$$

$$\sin\left(\frac{1}{\frac{1}{2n\pi+\frac{\pi}{2}}}\right) = \sin\left(2n\pi+\frac{\pi}{2}\right) = 1$$

$\forall n.$

and since $\lim_{x\to 0} \sin\frac{1}{x} = L$ by assumption

we have

$f(c_n)$ in Thm (iii)

$$L = \lim_{n\to\infty} \sin\left(\frac{1}{\frac{1}{n\pi}}\right) = 0$$

$$\| \qquad\qquad\qquad\qquad \neq$$

$$\lim_{n\to\infty} \sin\left(\frac{1}{\frac{1}{2n\pi+\frac{\pi}{2}}}\right) = 1$$

$(\rightarrow\leftarrow)$

So $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does NOT exist. #

§ Some important limits

Thm (§11.4)

(i) For $a>0$,

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

Since $f(x)=a^{x}$ is continuous,

$$\lim_{n\to\infty} a^{\frac{1}{n}} = a^{\lim\limits_{n\to\infty}\frac{1}{n}} = a^{0} = 1.$$

(ii) For $|a|<1$,

$$\lim_{n\to\infty} a^{n} = 0$$

Pf

$$\forall \varepsilon>0,\ \exists K=\max\Big\{1,\ \underset{\text{explain later}}{\underline{\underline{\overset{\overset{\log_{|a|}(\frac{\varepsilon}{2})}{\geq}}{\log_{|a|}(\tfrac{\varepsilon}{2})}}}}\Big\}>0$$

when $n\geq K$,

$$|a^n - 0| = |a|^n$$

($|a|<1$ $\Rightarrow$ $|a|^x$ is decreasing)

$$\leq |a|^K \leq |a|^{\log_{|a|}(\frac{\varepsilon}{2})}$$

$$= \frac{\varepsilon}{2} < \varepsilon \qquad \#$$

(iii) For $a>0$,

$$\lim_{n\to\infty} \frac{1}{n^a} = 0$$

i.e. $\lim_{x\to 0^+} x^a = 0^a = 0$

Since x^a is right continuous at 0

and $\overset{>0}{\frac{1}{n}} \longrightarrow 0$ as $n\to\infty$

we have

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^a = \left(\lim_{n\to\infty} \frac{1}{n}\right)^a = 0^a = 0$$

(iv) For $a \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

pf

Notation:

For $n > |a|$,

$$\left|\frac{a^n}{n!} - 0\right| = \frac{|a|^n}{n!}$$

$$= \overbrace{\left(\frac{|a|}{1}\right)\cdot\left(\frac{|a|}{2}\right)\cdot\left(\frac{|a|}{3}\right)\cdots\left(\frac{|a|}{n}\right)}^{n \text{ times}}$$

$[|a|]$

= the largest integer which is smaller than $|a|$

e.g.

$[1.234] = 1$

$[3.14] = 3$

$[e] = 2$

call it $\hat{a}$

$$= \underbrace{\boxed{\left(\frac{|a|}{1}\cdot\frac{|a|}{2}\cdots\frac{|a|}{[|a|]}\right)}}_{>1,\ \text{independent of } n\,!!}\cdot\left(\underbrace{\left(\frac{|a|}{[|a|]+1}\right)}_{\overset{\wedge}{1}}\underbrace{(\cdots)}_{\overset{\wedge}{1}}\underbrace{\left(\frac{|a|}{n}\right)}_{\overset{\wedge}{1}}\right)$$

$$\le \hat{a}\cdot\frac{|a|}{n}$$

So

$$\underbrace{-\hat{a}\cdot\frac{|a|}{n}}_{\downarrow\ 0 \text{ as } n\to\infty} \le \frac{a^n}{n!} \le \underbrace{\hat{a}\cdot\frac{|a|}{n}}_{\downarrow\ 0 \text{ as } n\to\infty}$$

$\forall\ n > |a|$

So, by Pinching Thm

$$\ell \quad \frac{a^n}{\quad} = 0 \qquad \forall$$

$$\lim_{n\to\infty} \quad n! \quad = 0 \quad \#$$

(v) $$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

pf

Consider

$$f(x) = \frac{\boxed{\ln x} \to \infty}{\boxed{x} \to \infty} \quad \text{as } x \to \infty \qquad \text{"}\frac{\infty}{\infty}\text{"}$$

By L'Hôpital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

So

$$f(n) = \frac{\ln n}{n} \to 0 \quad \#$$

(vi) $$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

pf:

Consider

$$f(x) = x^{\frac{1}{x}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\overset{\frac{\ln x}{x}}{\overset{\parallel}{\boxed{\ln(x^{\frac{1}{x}})}}}}$$

$$\overset{e^x \text{ is continuous}}{=} e^{\boxed{\lim_{x\to\infty}\frac{\ln x}{x}} = 0} = e^{0} = 1$$

So

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \sqrt[n]{n} = 1 \quad \#$$

Thm (§11.4)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

In particular $(x=1)$,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

pf

1

for 'x = 0,
left-hand-side

$$\text{LHS} = \lim_{n\to\infty} \overset{//}{\boxed{\left(1+\frac{0}{n}\right)^n}} = 1 = e^0$$

For $x \neq 0$,

$$\text{LHS} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n}$$

$$= e^{\boxed{\lim_{n\to\infty} n\cdot\ln\left(1+\frac{x}{n}\right)}} \overset{?}{=} \underline{\underline{x}}$$

Recall

$$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

$f(t) = \ln t$
$h = \frac{x}{n}$
$t = 1$

Here,

$$\lim_{n\to\infty} n\cdot\ln\left(1+\frac{x}{n}\right) = \lim_{n\to\infty} x\cdot\boxed{\frac{\ln\left(1+\frac{x}{n}\right)-\ln 1}{\frac{x}{n}}}$$

$$= x\cdot\left(\frac{d}{dt}\ln t\right)_{t=1} = x\,\frac{1}{t}\Big|_{t=1} = \underline{\underline{x}}$$

So

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \qquad \#$$

$$\left(\frac{d}{dt}\ln t\right)_{t=1} = \lim_{h\to 0}\boxed{\frac{\ln(1+h)-\ln 1}{h}}$$

i.e. $\lim_{h\to 0} f(h)$

Consider $$C_n = \frac{x}{n} \longrightarrow 0$$

So $$\lim_{n\to\infty} f\left(\frac{x}{n}\right) = \left(\frac{d}{dt}\ln t\right)\Big|_{t=1}$$

§ Relative rate of growth

Def

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be seq. s.t.

$$a_n, b_n > 0$$

for n sufficiently large.

(i) We say $(a_n)_{n=1}^{\infty}$ grows faster than $(b_n)_{n=1}^{\infty}$ (as $n\to\infty$) if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Or we can say $(b_n)_{n=1}^{\infty}$ grows slower than $(a_n)_{n=1}^{\infty}$

(ii) We say $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ grow at the same rate if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L > 0.$$

Example

① $(e^n)_{n=1}^{\infty}$ grows faster than $(n^2)_{n=1}^{\infty}$

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$

because

$$\lim_{n\to\infty} \frac{n^2}{e^n} = 0 = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

② $(3^n)_{n=1}^{\infty}$ grows faster than $(2^n)_{n=1}^{\infty}$

because

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

③ $(n^3+n)_{n=1}^{\infty}$ grow faster than $(n^2)_{n=1}^{\infty}$ n^3

because n^3

$$\lim_{n\to\infty} \frac{n^2 \times \frac{1}{n^3}}{(n^3+n) \times \frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1} = 0 \qquad 1 \qquad 1 \neq 0$$

④ $(n^2)_{n=1}^{\infty}$ grows faster than $(\ln n)_{n=1}^{\infty}$

because

$$\lim_{n\to\infty} \frac{\ln n}{n^2} \underset{\uparrow}{=} 0$$

$$\lim_{x\to\infty} \frac{\ln x^{\to\infty}}{x^2_{\to\infty}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = 0$$

⑤ $(n^{\frac{1}{k}})_{n=1}^{\infty}$ grows faster than $(\ln n)_{n=1}^{\infty}$

because

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt[k]{n}} \underset{\uparrow}{=} 0$$

$$\lim_{x\to\infty} \frac{\overset{\to\infty}{\ln x}}{\underset{\searrow\infty}{x^{\frac{1}{k}}}} = \lim_{x\to\infty} \frac{\frac{1}{\cancel{x}}}{\frac{1}{k}\cdot x^{\frac{1}{k}\cancel{-1}}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{k}\cdot\sqrt[k]{x}} = 0$$

slow fast

$$\dots, \ln n, \quad \sqrt[k]{n}, \quad n^k, \quad e^n, \quad e^{e^n}, \dots$$

<u>Def</u>

Let a_n, b_n be positive for n sufficintly lage

(i) $(a_n)_{n=1}^{\infty}$ is of smaller order than $(b_n)_{n=1}^{\infty}$ (as $n\to\infty$) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0,$$

denoted $a_n = o(b_n)$

"a_n is little-oh of b_n"

(ii) $(a_n)_{n=1}^{\infty}$ is of at most the order of $(b_n)_{n=1}^{\infty}$ (as $n\to\infty$) if $\exists$ M s.t.

$$\frac{a_n}{b_n} \le M \qquad \forall n,$$

denoted $a_n = O(b_n)$

"a_n is big-oh of b_n"

Example

① $\ln n = o(n)$ (i.e. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$)

② $n^2 = o(n^3+1)$

③ $n+\sin n = O(n)$

④ $e^n + n^2 = O(e^n)$

⑤ $n = o(e^n)$.

Example (Algorithms of searching)

Q: Find a certain word in a dictionary
(Design algorithms.) $\boxed{w}$
Assume there are n words in the dictionary

Algorithm 1: Check each word one by one.

The the worst case:

$\boxed{w}$ is the last word we check
We need n steps to find $\boxed{w}$
We say the time Complexity of Algorithm 1 is $O(n)$.

Algorithm 2 (binary search algorithm):

Know: words are in alphabet order in a dictionary.

Method: Open a page in the middle
(divide words into

↓

Check $\boxed{W}$ is in the first group 2 groups)
or the second group

↓

Repeat this process to the group with $\boxed{W}$

Worst case:

By one step, you can rule out one half of words.

By k steps, your target is in

$$\frac{n}{2^k}$$

worst case is need k steps,

$$2^k > n$$

i.e. $k > \log_2 n$ Need $\log_2 n$ steps

Time complexity of Algorithm2 is $O(\log_2 n)$

$$O(n) \longleftrightarrow O(\log_2 n)$$

$$\log_2 n = o(n)$$

This means, as $n \to \infty$, Algorithm 1 needs more steps, i.e. takes more time.

§ Improper integrals

Recall: $a, b \neq 0$

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$, i.e.

$$\int_a^b f(x)\,dx$$

exists.

Today: What if f is NOT continuous on $[a, b]$.? or a or $b = \infty$?

e.g. $\frac{1}{x}$ is NOT continuous $[0, 1]$

$\int_0^1 \frac{1}{x} dx = ?$

This Type of integrals are called improper integrals.

瑕積分

Def (§11.7)

Integrals with infinite upper/lower bounds are called improper integrals. (of type I)

eg. $\int_1^\infty \frac{1}{x} dx$

(i) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

(ii) If $f(x)$ is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

(iii) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

where c is any real number.

In each case, if the limit exists and finite, we say the improper integral converges and the limit is the value of the proper integral.

If the limit does NOT exist, we say the improper integral diverges.

Example

① $$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \boxed{\int_1^b \frac{\ln x}{x^2}\,dx}$$

Recall: $$\int_a^b u(x)\cdot v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)\cdot v(x)\,dx$$

"integration by parts"

$$u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$$

$$v(x) = -\frac{1}{x} \Rightarrow v'(x) = \frac{1}{x^2} = x^{-2}$$

$= -x^{-1}$ $\qquad -(-1)\cdot x^{-1-1}$ "

$(x^r)' = r\cdot x^{r-1}$

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x}\Big|_1^b + \int_1^b +\frac{1}{x^2}\,dx$$

$$= -\frac{\ln b}{b} - \left(-\overset{0}{\cancel{\frac{\ln 1}{1}}}\right) + \left(-\frac{1}{x}\right)\Big|_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

So

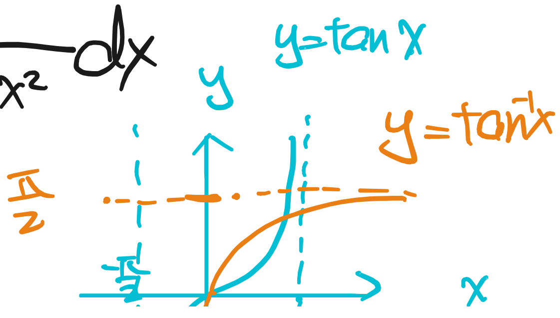
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\overset{0}{\cancel{\frac{\ln b}{b}}} - \overset{0}{\cancel{\frac{1}{b}}} + 1\right)$$

$$= 1 \qquad \#$$

② $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

Recall



$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

$$= \lim_{b\to\infty}\left(\tan^{-1}x \Big|_0^b\right) = \lim_{b\to\infty} \tan^{-1} b$$

$$= \frac{\pi}{2}$$

Similarly,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}dx$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_a^0 = \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a)$$

$$= -\lim_{a\to-\infty} \tan^{-1} a = -(-\frac{\pi}{2}) = \frac{\pi}{2}$$

So

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

\#

Thm

$$\int^{\infty} \frac{1}{\quad} dx \begin{cases} = \frac{1}{p-1}, & p>1 \end{cases}$$

$\int_1 \quad x^p$

$\Big($ diverges, $\quad p \le 1$

Pf

① For $p>1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty}\left(\underbrace{\frac{x^{-p+1}}{-p+1}\Bigg|_1^b}_{\frac{b^{-p+1}}{-p+1}-\frac{1}{-p+1}}\right)$$

$$= \lim_{b\to\infty}\left(\frac{1}{p-1} + \cancelto{0}{\frac{b^{\boxed{-p+1}}}{-p+1}}\right) \quad <0$$

$$= \frac{1}{p-1}$$

$$\ln x\Big|_1^b = \ln b$$

For $p=1$,

$$\int_1^\infty \frac{1}{x}\,dx = \lim\overset{\|}{\left(\int_1^b \frac{1}{x}\,dx\right)}$$

$\int_1 \quad x \qquad \lim_{b\to\infty} \int_1 \quad x$

$$= \lim_{b\to\infty} \ln b \quad \text{diverges}$$

For $p<1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\frac{1}{p-1} + \overset{\infty}{\frac{b^{\overset{>0}{-p+1}}}{-p+1}}\right)$$

diverges. #