Calculus, week 15, Spring 2025

Recall (Green's Thm)

$P = P(x,y)$

$Q = Q(x,y)$

$\partial\Omega$ = boundary of Ω

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \underbrace{\oint_{\partial\Omega} P\,dx + Q\,dy}_{\uparrow}$$

if $\partial\Omega$ is parametrized by $\vec{\gamma}(t) \overset{(\gamma_1(t),\, \gamma_2(t))}{=}$, $t \in [a,b]$

(counter-clockwise)

then
$$\oint_{\partial\Omega} P\,dx + Q\,dy$$

$$= \int_a^b P(\vec{\gamma}(t))\, \underline{d\gamma_1}_{\overset{\gamma_1'(t)\,dt}{\parallel}} + Q(\vec{\gamma}(t))\, \underline{d\gamma_2}_{\overset{\gamma_2'(t)\,dt}{\parallel}}$$

$$= \int_a^b \begin{pmatrix} P(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t) \\ + \; Q(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t) \end{pmatrix} dt$$

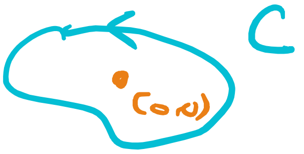
T |

Example

Let C be Jordan curve that does not pass through the origin (0,0). Show that

$$\oint_C \underbrace{-\frac{y}{x^2+y^2}}_{=P(x,y)}\,dx + \overbrace{\frac{x}{x^2+y^2}}^{=Q(x,y)}\,dy$$

$$= \begin{cases} 0 & \text{if } C \text{ does not enclose } (0,0) \\ 2\pi & \text{if } C \text{ encloses } (0,0) \end{cases}$$



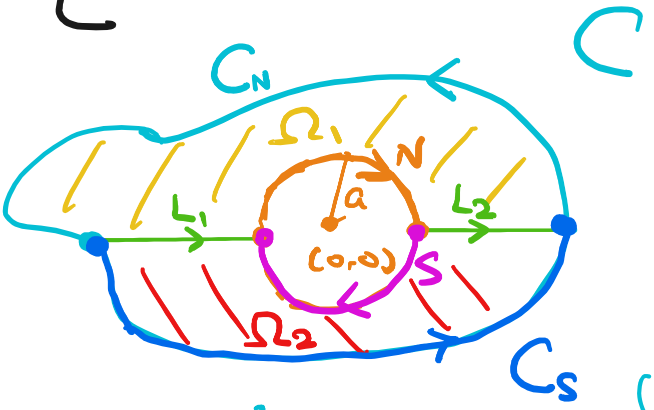
sol

① Assume C encloses (0,0).

We need a geometric fact: ∃ a circle

$$C_a:\ x^2+y^2=a^2$$

lies inside C



$(x^2+y^2) - x\cdot 2x$

$(x^2+y^2) - y\cdot 2y$

Note that

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = \overset{\frac{}{(x^2+y^2)^2}}{\overset{\shortparallel}{\underline{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)}}} + \overset{(x^2+y^2)}{\overset{\shortparallel}{\underline{\frac{\partial}{\partial y}\left(+\frac{y}{x^2+y^2}\right)}}}$$

$$= \begin{cases} \text{doesn't exist} & (x,y)=(0,0) \\ 0 & (x,y)\neq(0,0) \end{cases}$$

By Green's Thm,

(i) $$\iint_{\Omega_1} \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dxdy = 0$$

$$= \oint_{\partial\Omega_1} P\,dx + Q\,dy$$

$$= \underline{\int_{L_1} Pdx + Qdy} + \int_N Pdx + Qdy$$

$$+ \underline{\int_{L_2} Pdx + Qdy} + \int_{C_N} Pdx + Qdy$$

(ii) $$\iint_{\Omega_2} \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dxdy = 0$$

$$= \oint_{\partial\Omega_2} P\,dx + Q\,dy$$

$$= \underline{-\int_{L_1} - \int_{L_2}} + \int_S + \int_{C_S}$$

So (i) + (ii):

$$0 = \underbrace{\int_N + \int_S}_{\oint_{C_a} + \oint_C} + \int_{C_N} + \int_{C_S}$$

$$\Rightarrow \oint_C = -\oint_{C_a} = \oint_{C_a} P\,dx + Q\,dy$$

Finally, compute $\oint_{C_a} P\,dx + Q\,dy$:

$$\vec{\gamma}(t) = (a\cos t, a\sin t), \quad t \in [0, 2\pi]$$

is a parametrization of C_a

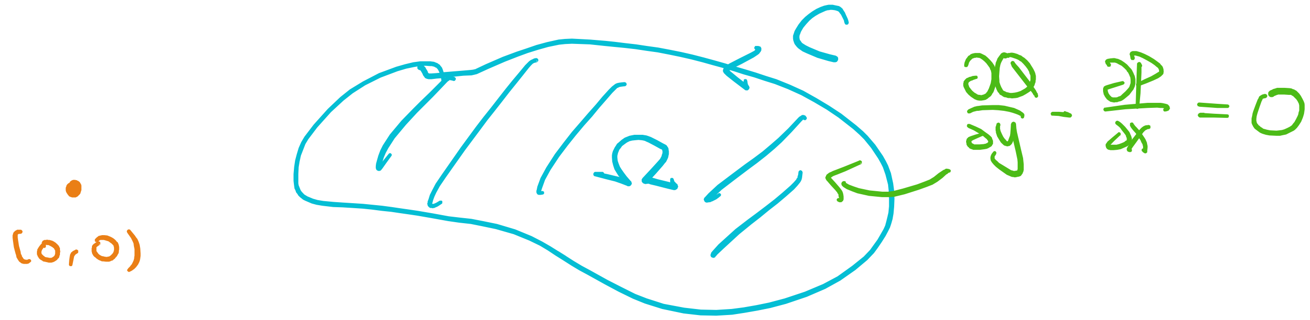
$$\Rightarrow \oint_{C_a} = \int_0^{2\pi} \left(P(a\cos t, a\sin t)\,(a\cos t)' + Q(a\cos t, a\sin t)\,(a\sin t)' \right) dt$$

$$a^2\sin^2 t + a^2\cos^2 t = a^2$$

$$= \int_0^{2\pi} \left(\frac{-a\sin t}{\underbrace{a^2\cos^2 t + a^2\sin^2 t}_{=a^2}} (-a\sin t) + \frac{a\cos t}{\underbrace{a^2\cos^2 t + a^2\sin^2 t}_{=a^2}} \, a\cos t \right) dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi = \oint_C P\,dx + Q\,dy$$

② If C doesn't enclose $(0,0)$, then by Green's Thm,



$$\oint_C P\,dx + Q\,dy = \iint_\Omega 0 \, dx\,dy = 0$$

\#

Remark

See Project 18.4 for an application to electric fields.

Divergence thm:

$$\iiint_T \left(\underline{\operatorname{div}\vec{v}}\right)^{=?} dx\,dy\,dz = \iint_{\partial T} \vec{v}\bullet\underline{\vec{n}}^{\,=?}\;\underset{=?}{\underline{d\sigma}}$$

Def

Let

$$S:\ \vec{r}(u,v) = r_1(u,v)\,\vec{i} + r_2(u,v)\,\vec{j} + r_3(u,v)\,\vec{k} \qquad (u,v)\in\Omega\subseteq\mathbb{R}^2$$

be a smooth parametrized surface.

Define the surface integral of H over S is

$$\iint_S H(x,y,z)\,d\sigma$$

$$= \iint_\Omega H(\vec{r}(u,v))\cdot\|\vec{N}(u,v)\|\,du\,dv$$

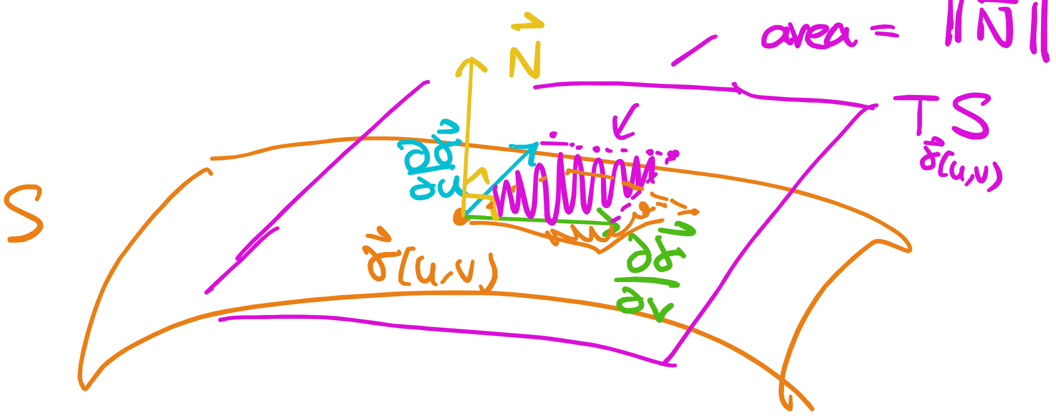
where

$$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \end{pmatrix} \qquad \partial\vec{r}$$

$$\vec N(u,v) = \det\begin{pmatrix} \frac{\partial\gamma_1}{\partial u} & \frac{\partial\gamma_2}{\partial u} & \frac{\partial\gamma_3}{\partial u} \\ \frac{\partial\gamma_1}{\partial v} & \frac{\partial\gamma_2}{\partial v} & \frac{\partial\gamma_3}{\partial v} \end{pmatrix}$$

$\frac{\partial\vec\gamma}{\partial u}$

$\frac{\partial\vec\gamma}{\partial v}$



$$= \frac{\partial\vec\gamma}{\partial u} \times \frac{\partial\vec\gamma}{\partial v}$$

$$= \left(\frac{\partial\gamma_2}{\partial u}\frac{\partial\gamma_3}{\partial v} - \frac{\partial\gamma_3}{\partial u}\frac{\partial\gamma_2}{\partial v}\right)\vec i$$

$$- \left(\frac{\partial\gamma_1}{\partial u}\frac{\partial\gamma_3}{\partial v} - \frac{\partial\gamma_3}{\partial u}\frac{\partial\gamma_1}{\partial v}\right)\vec j$$

$$+ \left(\frac{\partial\gamma_1}{\partial u}\frac{\partial\gamma_2}{\partial v} - \frac{\partial\gamma_2}{\partial u}\frac{\partial\gamma_1}{\partial v}\right)\vec k$$

<u>Example</u>

Let

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

be the unit sphere.

$\iint 1 \, d\sigma = ?$ (= surface area)

$\iint_S 1 \, dS = ?$ (surface area of S)

Sol

Note that

$$S = S_{\geq 0} \cup S_{<0}$$

where

$$S_{\geq 0} = \{(x,y,z) \mid x^2+y^2+z^2=1, \ z \geq 0\}$$

$$S_{<0} = \{(x,y,z) \mid x^2+y^2+z^2=1, \ z < 0\}$$

$\Rightarrow$

(i) $\vec{r}(u,v) = u\vec{i} + v\vec{j} + \sqrt{1-u^2-v^2} \ \vec{k}$,

$u,v \in \bar{D} = \{ u^2+v^2 \leq 1 \}$

parametrizes $S_{\geq 0}$.

(ii) $\vec{\beta}(u,v) = u\vec{i} + v\vec{j} - \sqrt{1-u^2-v^2} \ \vec{k}$,

$u,v \in D = \{u^2+v^2 < 1\}$

parametrizes $S_{<0}$.

Note that:

(i) $$\vec{N}_\gamma = \det\begin{pmatrix} i & j & k \\ \frac{\partial u}{\partial u} & \frac{\partial v}{\partial u} & \frac{\partial\sqrt{1-u^2-v^2}}{\partial u} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial v} & \frac{\partial\sqrt{1-u^2-v^2}}{\partial v} \end{pmatrix}$$

$$= \det\begin{pmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ 1 & 0 & \frac{-u}{\sqrt{1-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

$$= \frac{u}{\sqrt{1-u^2-v^2}}\vec{i} + \frac{v}{\sqrt{1-u^2-v^2}}\vec{j} + \vec{k}$$

$$\Rightarrow \|\vec{N}_\gamma\| = \sqrt{\frac{u^2+v^2+(1-u^2-v^2)}{1-u^2-v^2}}$$

$$= \frac{1}{\sqrt{1-u^2-v^2}}$$

(ii) $$\vec{N}_\beta = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial u}{\partial u} & \frac{\partial v}{\partial u} & -\frac{\partial\sqrt{1-u^2-v^2}}{\partial u} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial v} & -\frac{\partial\sqrt{1-u^2-v^2}}{\partial v} \end{pmatrix}$$

$$= -\frac{u}{\sqrt{1-u^2-v^2}}\vec{i} - \frac{v}{\sqrt{1-u^2-v^2}}\vec{j} + \vec{k}$$

$$\Rightarrow \|\vec{N}_\beta\| = \frac{1}{\sqrt{1-u^2-v^2}}$$

Therefore,

$$\iint_{S} 1\, d\sigma = \iint_{S_{\geq 0}} 1\, d\sigma + \iint_{S_{<0}} 1\, d\sigma$$

$$= \iint_{\bar{D}} 1 \cdot \overset{= \frac{1}{\sqrt{1-u^2-v^2}}}{\boxed{\|\vec{N}_{\alpha}\|}}\, du\, dv$$

$u = r\cos\theta$

$v = r\sin\theta$

$\ni J(r,\theta) = r$

$$+ \iint_{D} 1 \cdot \overset{= \frac{1}{\sqrt{1-u^2-v^2}}}{\boxed{\|\vec{N}_{\beta}\|}}\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{1} \frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta$$

$$+ \int_0^{2\pi} \underbrace{\int_0^{1} \frac{1}{\sqrt{1-r^2}}\, r\, dr}_{= -(1-r^2)^{\frac{1}{2}}\Big|_{r=0}^{1} = 1} d\theta$$

$$= 2 \int_0^{2\pi} 1\, d\theta = 2 \cdot 2\pi = 4\pi$$

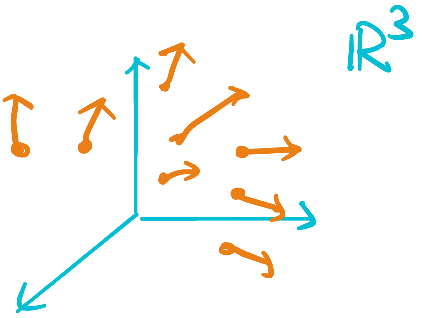
= surface area of  #

Def (18.8.2)

Let

$$\vec V(x,y,z) = V_1(x,y,z)\,\vec i + V_2(x,y,z)\,\vec j + V_3(x,y,z)\,\vec k$$

be a vector field on $\mathbb{R}^3$, that is, a triple of smooth functions on $\mathbb{R}^3$.



散度

The divergence of $\vec V$ is the function

$$\operatorname{div}\vec V = \nabla\bullet\vec V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

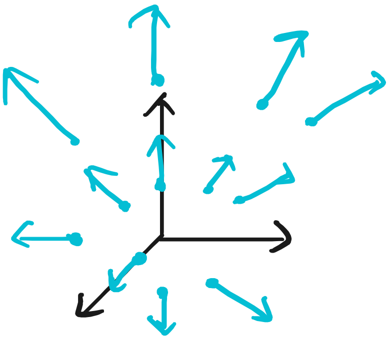
旋度

The curl of $\vec v$ is the vector field

$$\operatorname{curl}\vec v = \nabla\times\vec V$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec i + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec k$$

Example



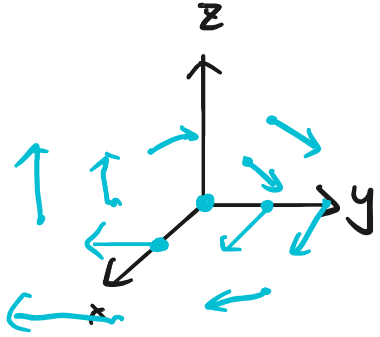
① $\vec V = x\vec i + y\vec j + z\vec k$

$$\Rightarrow \operatorname{div}\vec V = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\operatorname{curl} \vec{v} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\vec{j}$$

$$+\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\vec{k} = \vec{0}$$

② $\vec{V} = y\vec{i} - x\vec{j}$



$$\operatorname{div} \vec{V} = \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} + \frac{\partial 0}{\partial z} = 0$$

$$\operatorname{curl} \vec{V} = \left(\frac{\partial 0}{\partial y} - \frac{\partial(-x)}{\partial z}\right)\vec{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial 0}{\partial x}\right)\vec{j}$$

$$+\left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right)\vec{k}$$

$$= -2\vec{k}$$

<u>Thm</u> (Divergence Thm, Thm 18.9.2)

Let T be a solid bounded by a piecewise smooth surface S.

If $\vec{v}$ is a smooth vector field on T,

then

$$\iiint_T \operatorname{div}\vec{v}\, dxdydz$$

$$= \iint_S \vec{v}\bullet\vec{n}\, d\sigma$$

"flux"

where $\underline{\vec{n} = \vec{n}(x,y,z)}$ is the outer unit normal vector of S.

If S is defined by $f(x,y,z) = c$, then $\nabla f(x,y,z)$ is a normal vector of S.

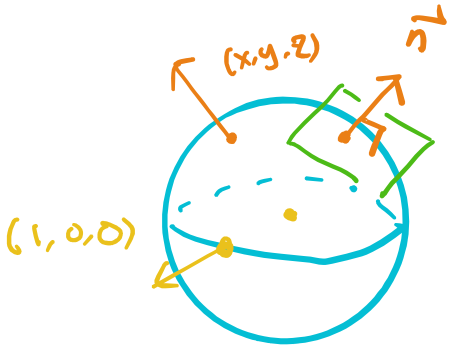
$$\Rightarrow \quad \vec{n} = \frac{\nabla f}{\|\nabla f\|} \text{ or } -\frac{\nabla f}{\|\nabla f\|}$$

Example

Let

$$B = \{x^2+y^2+z^2 \le 1\}$$

$$S = \partial B = \{x^2+y^2+z^2=1\}$$



① $\vec{n}$ = outer $\overset{\wedge}{\text{unit}}$ normal vector = ?

(x,y,z) ∈ S

$$\nabla(x^2+y^2+z^2) = (2x,\ 2y,\ 2z)$$

$$\frac{(2x,2y,2z)}{\|(2x,2y,2z)\|} = \frac{(2x,2y,2z)}{\sqrt{\underset{\underset{4}{\shortparallel}}{\boxed{4x^2+4y^2+4z^2}}}}$$

$$= (x,y,z)$$

So $\vec{n} = (x,y,z)$

② Consider the vector field

$$\vec{v} = (x,y,z)$$

$\Rightarrow$

$$\operatorname{div}(\vec{v}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\vec{v} \bullet \vec{n} = (x,y,z) \bullet (x,y,z)$$

$$= x^2+y^2+z^2 = 1 \quad \text{for } (x,y,z) \in S$$

By the divergence thm,

$$\iiint_B \overset{\overset{3}{\shortparallel}}{\underline{\operatorname{div}(\vec{V})}}\, dxdydz = \iint_S \overset{\overset{1}{\shortparallel}}{\underline{\vec{v} \bullet \vec{n}}}\, d\sigma$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$3 \operatorname{vol}(B) \qquad\qquad\qquad \operatorname{area}(S)$$

$3\ \mathrm{vol}(B) \qquad\qquad\qquad \mathrm{area}(S)$

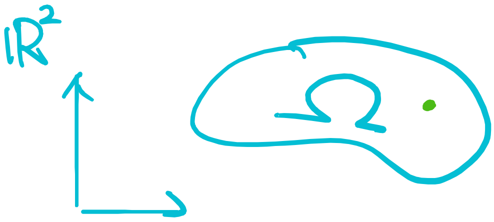
$\qquad \| \qquad\qquad\qquad\qquad \|$

$3\cdot\frac{4}{3}\pi \qquad\qquad\qquad 4\pi$

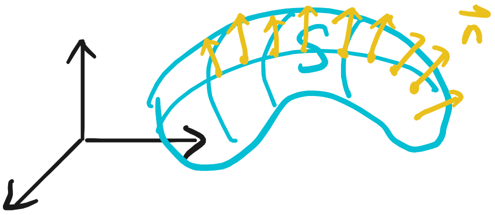
#

<u>Recall</u> (Green's Thm)

$$\iint_{\Omega}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \oint_{\partial\Omega}\underbrace{P\,dx+Q\,dy}_{\underline{(P,Q)}\cdot\underline{d\mathbf{r}}}$$



what if we consider



<u>Thm</u> (Stokes Thm, Thm 18.10.1)

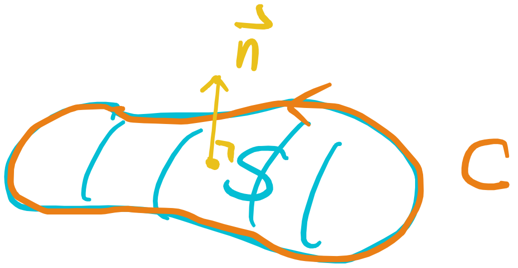
We are given a unit normal vector $\vec{n}$ on S

Let S be a smooth <u>oriented</u> surface with a (piecewise) smooth bounding curve C in $\mathbb{R}^3$.
If $\vec{v}$ is a smooth vector field on $\mathbb{R}^3$, then

$$\iint_S (\operatorname{curl} v) \bullet n \, d\sigma$$

$$= \int_C \underbrace{\vec{v}(r) \bullet dr}_{\overset{\shortparallel}{v_1\,dx + v_2\,dy + v_3\,dz}}$$

where $\int_C$ is taken in the positive sense with respect to $\vec{n}$



Example

Let

$$\vec{v} = (z^2, -2x, y^3)$$

$$S = \{ (x,y,z) \in \mathbb{R}^2 \mid x^2+y^2+z^2=1, \ z \geq 0 \}$$

$$\vec{\gamma}(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

Then

① $$\operatorname{curl}\Big(\overset{\overset{(z^2,\,-2x,\,y^3)}{\shortparallel}}{\vec{v}}\Big) = \left(\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right), \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right), \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \right)$$

$$= \left(3y^2 - 0,\ 2z - 0,\ -2 - 0 \right)$$

$$= \left(3y^2,\ 2z,\ -2 \right)$$

② Take

$$\vec{n} = (x, y, z)$$

$$\Rightarrow \quad \mathrm{Curl}(\vec{v}) \bullet \vec{n} = 3y^2 \cdot x + 2z \cdot y - 2 \cdot z$$

$$\Rightarrow \iint_S (\mathrm{Curl}\,\vec{v}) \bullet \vec{n}\, d\sigma$$

$$= \iint_S 3xy^2 + 2yz - 2z\, d\sigma$$

③ S can be parametrized by

$$\vec{\gamma}(u,v) = \left(u, v, \sqrt{1-u^2-v^2} \right).$$

$$(u,v) \in \bar{D} = \{ u^2 + v^2 \leq 1 \}$$

$$\Rightarrow \quad \iint_S 3xy^2 + 2yz - 2z \quad d\sigma$$

$$\frac{1}{\sqrt{1-u^2-v^2}}$$

11 by Tuesday

$$= \iint_{\bar{D}} \left(3uv^2 + 2v\sqrt{1-u^2-v^2} - 2\sqrt{1-u^2-v^2}\right) \boxed{\|\vec{N}_\gamma\|}\, du\, dv$$

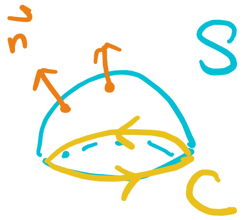
$$\frac{3(-u)v^2}{\sqrt{1-(-u)^2-v^2}} = -\frac{3uv^2}{\sqrt{1-u^2-v^2}}$$

← odd function in u

$$= \underbrace{\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{3uv^2}{\sqrt{1-u^2-v^2}}\, du\, dv}_{= 0} + \underbrace{\iint_{\bar{D}} 2v\, du\, dv}_{= \int_{-1}^{1}\underbrace{\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 2v\, dv}_{= 0}\, du} - \iint_{\bar{D}} 2\, du\, dv$$

$$= -2\iint_{\bar{D}} du\, dv = -2\ \text{area}(\bar{D}) = \underset{\iint_S (\text{curl}\,\vec{v})\cdot\vec{n}\, d\sigma}{\underset{\parallel}{-2\pi}}$$

④ $\int_C \vec{v}(r)\cdot dr = -2\pi$?



C can be parametrized by

$$\underbrace{(\cos\theta,\ \sin\theta,\ 0)}\qquad \theta \in [0, 2\pi]$$

$\vec{v} = (z^2, -2x, y^3)$

$$\Rightarrow \int_C \vec{v}(r)\cdot dr = \int_0^{2\pi} \left(\vec{v}(\cos\theta, \sin\theta, 0) \right) d\theta$$

$$\bullet (-\sin\theta, \cos\theta, 0))$$

$$= \int_0^{2\pi} (0, -2\cos\theta, \sin^3\theta) \bullet (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} -2\cos^2\theta \, d\theta = -2 \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} d\theta$$

$$= -2 \cdot \frac{1}{2} \cdot 2\pi = -2\pi \qquad \#$$

Remark

If S doesn't have boundary.

e.g. $\{x^2+y^2+z^2=1\}$

then

$$\iint_S (\text{curl}\, \vec{v}) \bullet \vec{n}\, d\sigma = 0$$

Remark

Green's Thm, Stoke's Thm, Divergence Thm can be unified: $\int_M d\omega = \int_{\partial M} \omega$

"Stoke's Thm in Differential Geometry"

by considering "differential forms."

$d\xi d\xi = 0$, $d\xi d\eta = - d\eta d\xi$

In particular, curl can be obtained by:

$$\vec{V} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

$$\leftrightarrow \quad v_1 dx + v_2 dy + v_3 dz$$

exterior derivative

$$\rightsquigarrow \underline{d}(v_1 dx + v_2 dy + v_3 dz)$$

$$= dv_1 dx + dv_2 dy + dv_3 dz$$

$$= \left(\cancel{\frac{\partial v_1}{\partial x} dx} + \frac{\partial v_1}{\partial y} dy + \frac{\partial v_1}{\partial z} dz\right) \underline{dx}$$

$$+ \left(\frac{\partial v_2}{\partial x} dx + \cancel{\frac{\partial v_2}{\partial y} dy} + \frac{\partial v_2}{\partial z} dz\right) \underline{dy}$$

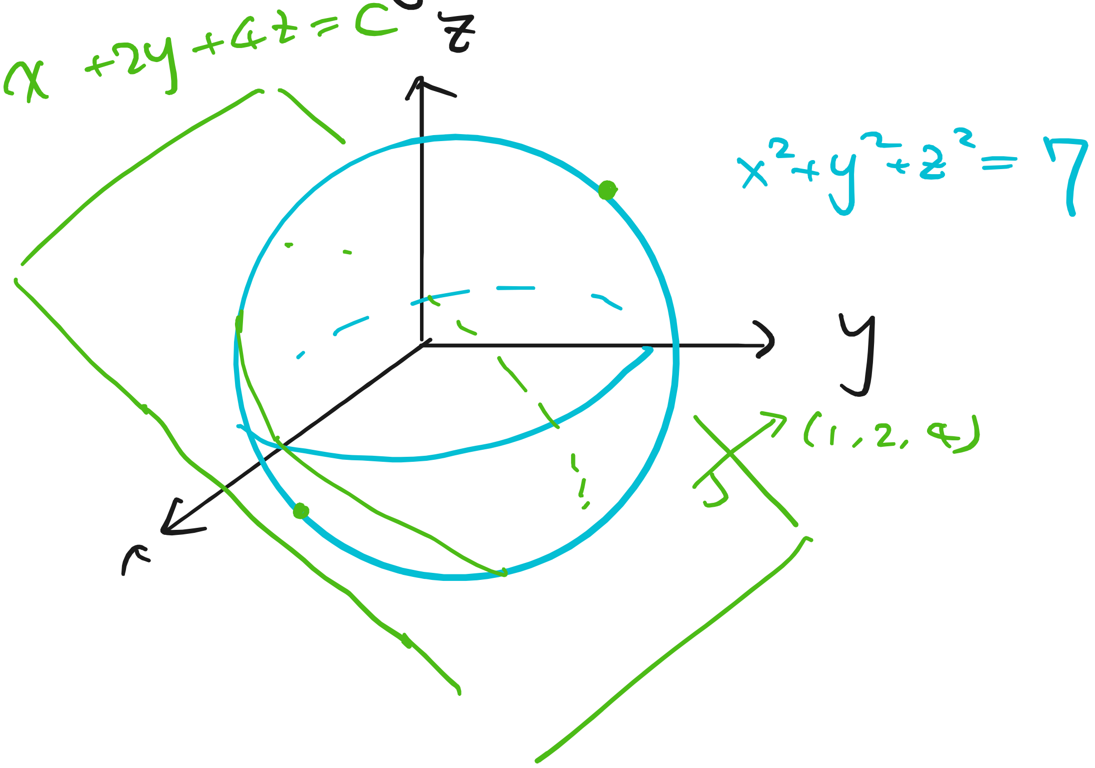
$$+ \left(\frac{\partial v_3}{\partial x} dx + \frac{\partial v_3}{\partial y} dy + \cancel{\frac{\partial v_3}{\partial z} dz}\right) \underline{dz}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) dy dz + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) dz dx$$

$$+ \left(\frac{\partial V_2}{\partial x} - \frac{\partial U_1}{\partial y} \right) dx\, dy$$

HW10.3.(c)

Minimize $\underline{x+2y+4z}^{=C}$ on $x^2+y^2+z^2=7$



Lagrange multiplier:

Extreme value of

$$g(x,y,z) = x+2y+4z$$

on

$$f(x,y,z) = x^2+y^2+z^2 = 7$$

occur only when

$$\begin{array}{ccc} \nabla g & // & \nabla f \\ \parallel & & \parallel \\ (1,2,4) & & (2x,2y,2z) \end{array}$$

$\Leftrightarrow \exists \lambda$ s.t.

$$\begin{array}{l} \lambda^2\left(\frac{1}{4}+1+4\right) = \frac{21}{4}\lambda^2 \\ \parallel \end{array}$$

$$\begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 4\lambda \\ x^2+y^2+z^2=7 \end{cases} \Rightarrow \left(\frac{\lambda}{2}\right)^2 + \lambda^2 + (2\lambda)^2 = 7$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{4}{3}}$$

$$\Rightarrow (x,y,z) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$$

$$\text{or } \left(-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$$

Then check which one of

$$g\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) \text{ and } \underset{\uparrow \text{ ans}}{\underline{g\left(-\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)}}$$

is smaller.

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