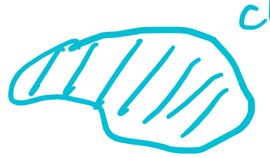
Recall

Last time, we said that a continuous on a closed disk or on a closed rectangle must attain a max. value and a min. value.

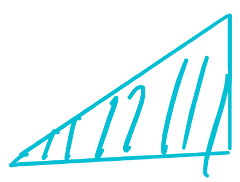
More generally, if K is "closed" and "bounded", e.g.



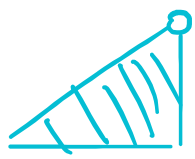
closed, bounded

NOT closed, bounded,

$\{|x| \leq 1\}$ closed, NOT bounded.



closed, bounded,



NOT closed

if f is continuous on K, then f attains a max value and a min. value on K.

Recall

$z = f(x,y)$

volume

$$= \iint_{\Omega} f(x,y)\,dx\,dy$$

double integral

Elementary classical analysis, 2ed.
Hoffman

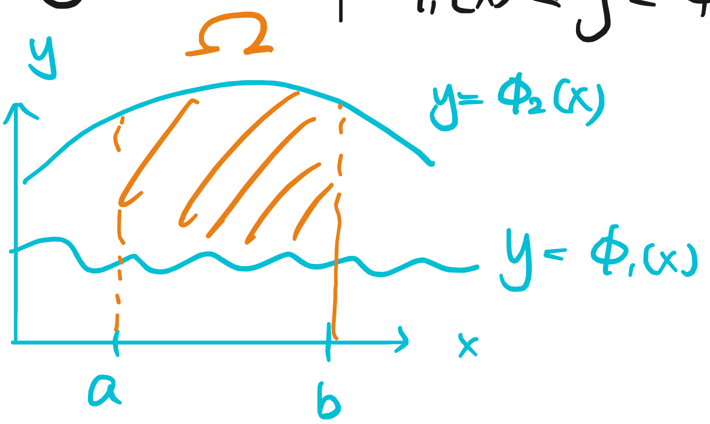
Thm (Marsden's book, Corollary 9.2.2)

Let $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ be continuous functions

s.t. $$\phi_1(x) \le \phi_2(x) \quad \forall\ x \in [a,b]$$

Let

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \le x \le b, \\ \phi_1(x) \le y \le \phi_2(x) \end{array} \right\}$$



If f is a continuous function on Ω

then

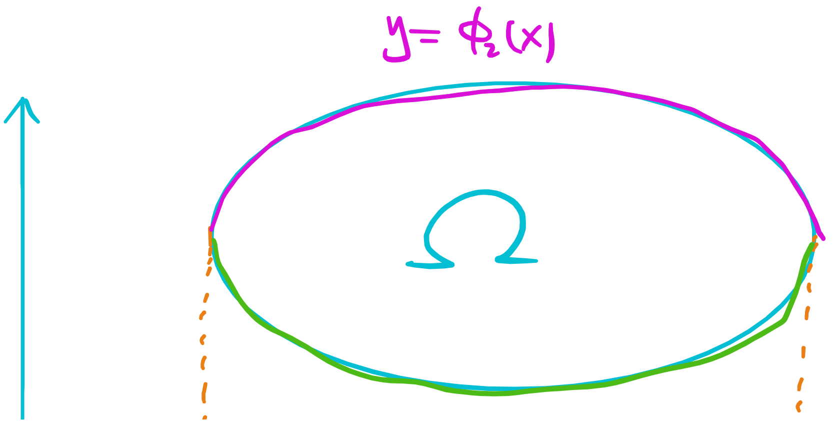
$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

Also see (17.3.1) and (17.3.2) in textbook.

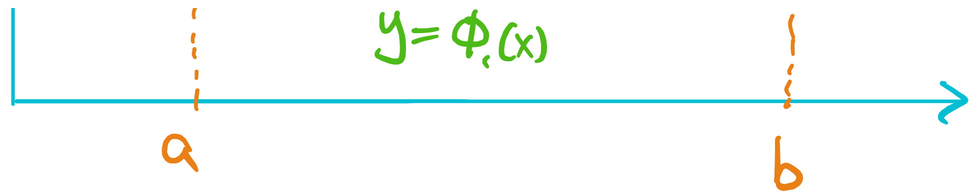
There are ways to compute $\iint_{\Omega} f(x,y)\,dx\,dy$.

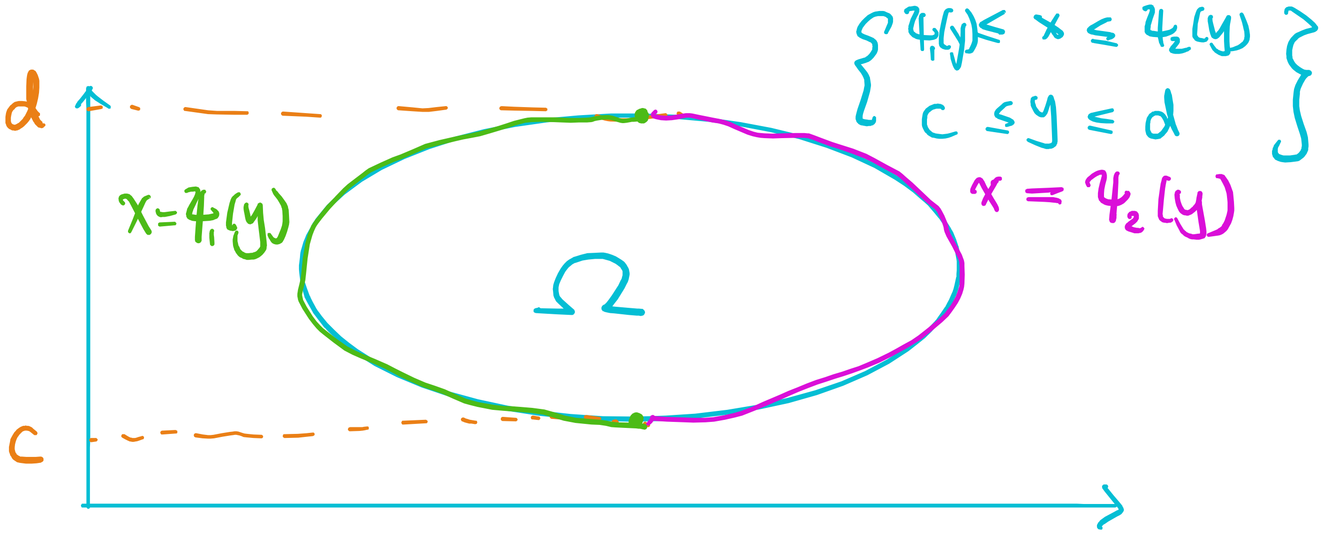
(i)
$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

(ii)
$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$



$$\left\{ \begin{array}{c} \phi_1(x) \le y \le \phi_2(x) \\ a \le x \le b \end{array} \right\}$$





<u>Example</u>

① $$\iint_R (x+y-2)\,dx\,dy,$$

$$R = \{\, 1 \le x \le 4,\quad 1 \le y \le 3 \,\}$$

<u>Sol</u>

<u>method I:</u>

$$\frac{\partial}{\partial y}\left(xy+\frac{y^2}{2}-2y\right)$$

$$\iint_R (x+y-2)\,dx\,dy = \int_1^4 \left(\int_1^3 \underline{(x+y-2)}\,dy \right) dx$$

$$= \int_1^4 \left. xy+\frac{y^2}{2}-2y \right|_{y=1}^{3} dx$$

$$3x+\frac{9}{2}-6-\left(x+\frac{1}{2}-2\right)=2x$$

$$=\int_1^4 2x\,dx = x^2\Big|_{x=1}^{4} = 16-1=15 \ \#$$

method II:

$$\frac{\partial}{\partial x}\left(\frac{x^2}{2}+xy-2x\right)$$

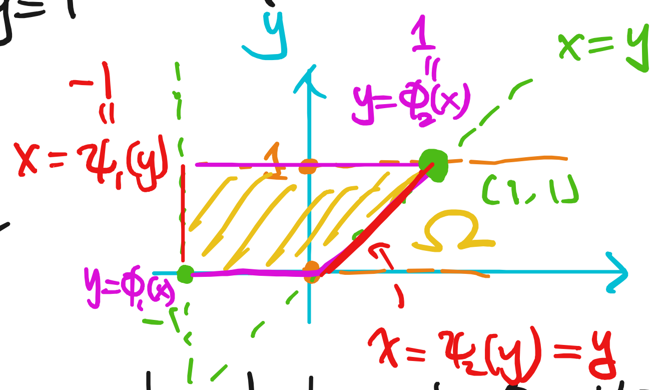
$$\iint_R x+y-2\,dx\,dy = \int_1^3\left(\int_1^4 x+y-2\,dx\right)dy$$

$$=\int_1^3 \underbrace{\frac{x^2}{2}+xy-2x\Big|_{x=1}^{4}}_{\frac{16}{2}+4y-8-\left(\frac{1}{2}+y-2\right)=3y+\frac{3}{2}} dy$$

$$\frac{\cancel{16}}{\cancel{2}}+4y\cancel{-8}-\left(\frac{1}{2}+y-2\right)=3y+\frac{3}{2}$$

$$=\frac{3}{2}y^2+\frac{3}{2}y\Big|_{y=1}^{3}=\frac{3}{2}(9+3-1-1)=15 \ \#$$

② $$\iint_\Omega xy-y^3\,dx\,dy$$



$$\phi_1(x)=\begin{cases}0, & x\le 0\\ x, & x\ge 0\end{cases}$$

Ω is the region enclosed by $y=0$, $y=1$, $x=-1$, $x=y$

sol

$$\begin{cases}0 & x\le 0\\ x & x\ge 0\end{cases}$$

method I:

$$\Omega = \{ \phi_1(x) \le y \le \phi_2(x) = 1 \ , \ -1 \le x \le 1 \}$$

$$\Rightarrow \iint_{\Omega} xy - y^3 \, dx\,dy = \int_{-1}^{1} \left(\int_{\phi_1(x)}^{1} xy - y^3 \, dy \right) dx$$

$$= \underbrace{\int_{-1}^{0} \left(\int_{\phi_1(x)}^{1} xy - y^3 \, dy \right)}_{} dx + \underbrace{\int_{0}^{1} \left(\int_{\phi_1(x)}^{1} xy - y^3 \, dy \right) dx}_{}$$

$\phi_1(x) = 0$ $\qquad\qquad$ $\phi_1(x) = x$

$$x\frac{y^2}{2} - \frac{y^4}{4} \Big|_{y=0}^{1} = \left(\frac{x}{2} - \frac{1}{4}\right) - 0$$

$$x\frac{y^2}{2} - \frac{y^4}{4} \Big|_{y=x}^{1} = \left(\frac{x}{2} - \frac{1}{4}\right) - \left(\frac{x^3}{2} - \frac{x^4}{4}\right)$$

$$= \int_{-1}^{0} \frac{x}{2} - \frac{1}{4} \, dx + \int_{0}^{1} \frac{x}{2} - \frac{1}{4} - \frac{x^3}{2} + \frac{x^4}{4} \, dx$$

$$= \frac{x^2}{4} - \frac{x}{4} \Big|_{-1}^{0} + \frac{x^2}{4} - \frac{x}{4} - \frac{x^4}{8} + \frac{x^5}{20} \Big|_{0}^{1}$$

$$= -\left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{4} - \frac{1}{8} + \frac{1}{20}\right) - 0$$

$$= -\frac{1}{2} \qquad - \frac{1}{8} + \frac{1}{20} = -\frac{23}{40}$$

#

<u>method II</u>:

$$\Omega = \{-1 \le x \le y,\quad 0 \le y \le 1\}$$

$$\iint_{\Omega} xy - y^3 \, dx\,dy = \int_0^1 \underbrace{\left(\int_{-1}^{y} xy - y^3 \, dx \right)}_{\parallel} dy$$

$$y \cdot \frac{x^2}{2} - y^3 \cdot x \Big|_{x=-1}^{y}$$

$$= \frac{y^3}{2} - y^4 - \left(\frac{y}{2} + y^3\right)$$

$$= -\frac{y}{2} - \frac{y^3}{2} - y^4$$

$$= \int_0^1 -\frac{y}{2} - \frac{y^3}{2} - y^4 \, dy$$

$$= -\frac{y^2}{4} - \frac{y^4}{2 \cdot 4} - \frac{y^5}{5} \Big|_{y=0}^{1}$$

$$= -\frac{1}{4} - \frac{1}{8} - \frac{1}{5} = -\frac{10+5+8}{40} = -\frac{23}{40} \quad \#$$

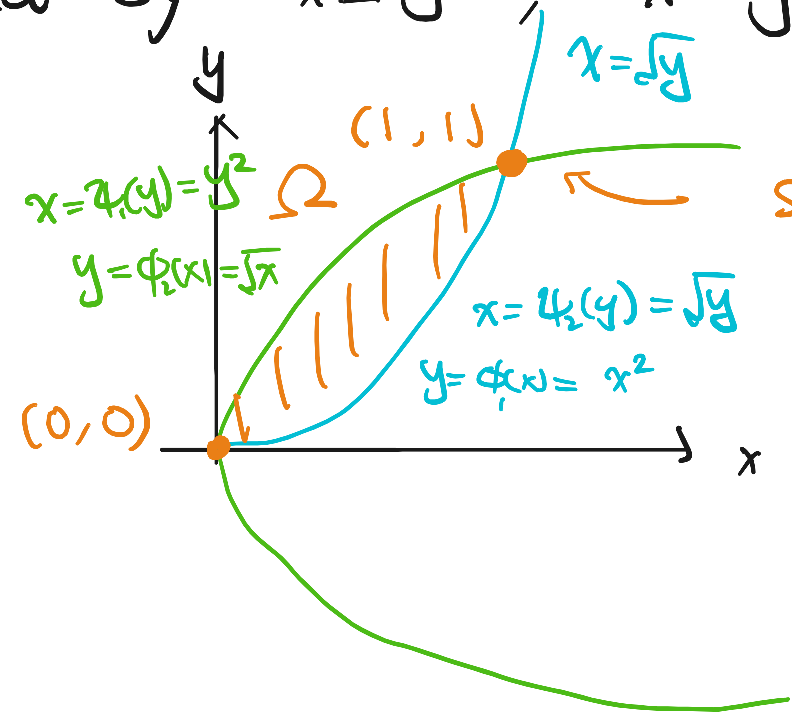
③ $\displaystyle\iint_{\Omega} x^{\frac{1}{2}} - y^2 \, dx\,dy,$

第一象限

Ω is the region, in the first quadrant,

enclosed by $x=y^{\frac{1}{2}}$, $x=y^2$

sol



Solve $\begin{cases} x=\sqrt{y} \\ x=y^2 \end{cases}$

$\Rightarrow y^2=\sqrt{y} \Rightarrow y=y^4$

$\Rightarrow y=0, 1$

method I

$$\Omega=\{\, x^2\le y\le\sqrt{x},\quad 0\le x\le 1 \,\}$$

$$\Rightarrow \iint_\Omega \sqrt{x}-y^2\,dx\,dy=\int_0^1\underbrace{\left(\int_{x^2}^{\sqrt{x}}\sqrt{x}-y^2\,dy\right)}_{\parallel}dx$$

$$\sqrt{x}\,y-\frac{y^3}{3}\Big|_{y=x^2}^{\sqrt{x}}=\sqrt{x}\cdot\sqrt{x}-\frac{x^{\frac{3}{2}}}{3}-\left(\sqrt{x}\cdot x^2-\frac{x^6}{3}\right)$$

$$=x-\frac{1}{3}\cdot x^{\frac{3}{2}}-x^{\frac{5}{2}}+\frac{1}{3}x^6$$

$$= \int_0^1 x - \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{5}{2}} + \frac{1}{3}x^6 \; dx$$

$$= \frac{x^2}{2} - \frac{1}{3}\,\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{1}{3}\,\frac{x^7}{7} \Bigg|_{x=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{3}\cdot\frac{2}{5} - \frac{2}{7} + \frac{1}{3}\cdot\frac{1}{7}$$

$$= \frac{105 - 28 - 60 + 10}{210} = \frac{27}{210} = \frac{9}{70} \quad \#$$

<u>method II</u>

$$\Omega = \{ y^2 \le x \le \sqrt{y},\; 0 \le y \le 1 \}$$

$$\iint_\Omega (\sqrt{x} - y^2)\, dx\,dy = \int_0^1 \left(\underbrace{\int_{y^2}^{\sqrt{y}} \sqrt{x} - y^2 \; dx}_{=\; \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - y^2 x \Big|_{x=y^2}^{\sqrt{y}}} \right) dy$$

$$= \frac{2}{3} y^{\frac{3}{4}} - y^{\frac{5}{2}} - \frac{2}{3}\cdot y^3 + y^4$$

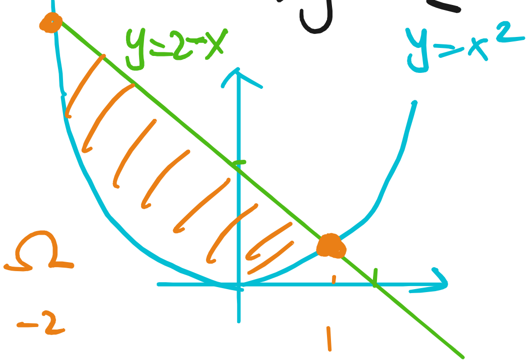
$$= \int_0^1 \frac{2}{3} y^{\frac{3}{4}} - y^{\frac{5}{2}} - \frac{2}{3} y^3 + y^4 \; dy$$

$$= \frac{2}{3} \cdot \frac{y^4}{\frac{7}{4}} - \frac{y^2}{\frac{7}{2}} - \frac{2}{3}\frac{y^4}{4} + \frac{y^5}{5} \Bigg|_0$$

$$= \frac{2}{3} \cdot \frac{4}{7} - \frac{2}{7} - \frac{1}{6} + \frac{1}{5}$$

$$= \frac{\overset{20}{\cancel{80 - 60}} \; \overset{+7}{\cancel{-35 + 4}}2}{210} = \frac{9}{70} \quad \#$$

④ Compute the area of the region Ω enclosed by $y = x^2$ and $x + y = 2$



<u>sol</u>

$$\text{Area} = \iint_{\Omega} 1 \, dx\,dy$$

$$= \int_{-2}^{1} \underbrace{\left(\int_{x^2}^{2-x} 1 \, dy \right)}_{2-x-x^2} dx$$

Solve

$$\begin{cases} y = 2 - x \\ y = x^2 \end{cases}$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1)$$

$$x = -2, 1$$

$$= \int^{1} 2 - x - x^2 \; d$$

$$= \int_{-2}^{1} 2 - x - x^2 \, dx$$

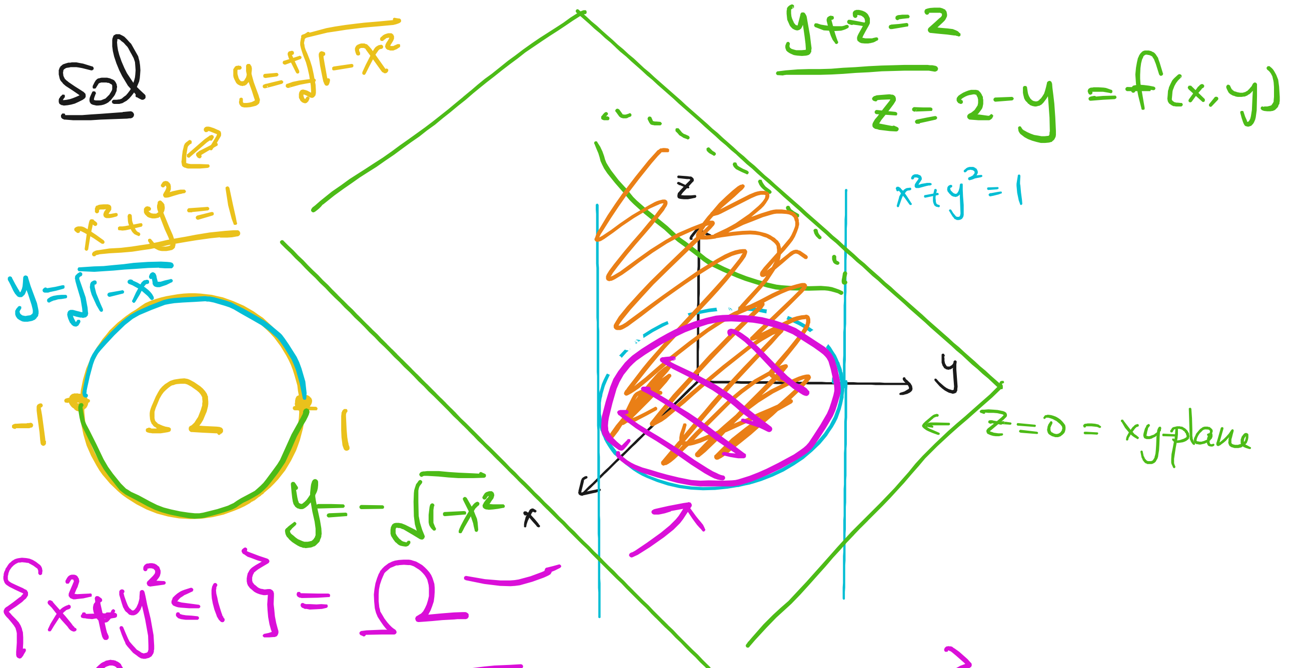
$$= 2x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{x=-2}^{1}$$

$$= 2 - \frac{1}{2} - \frac{1}{3} - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= \cancel{6}^{3} + \frac{3}{2} - \cancel{\frac{9}{3}} = \frac{9}{2} \quad \#$$

⑤ Calculate the volume within the cylinder $x^2+y^2=1$ between the planes $y+z=2$ and $z=0$

Sol



$\{x^2+y^2\le 1\} = \Omega$

$= \{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \; -1 \le x \le 1\}$

Volume = $\iint \;\; dx\,dy$

Volume $= \iint_{\Omega} 2-y \; dx\,dy$,

$$= \int_{-1}^{1} \underbrace{\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2-y \; dy \right)}_{=} dx$$

$$2y - \frac{y^2}{2} \Big|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 2\sqrt{1-x^2} - \cancel{\frac{1-x^2}{2}} - \left(-2\sqrt{1-x^2} - \cancel{\frac{1-x^2}{2}}\right)$$

$$= 4\sqrt{1-x^2}$$

$0 \le u \le \pi$

$x = \cos u$

$dx = -\sin u \, du$

$$= 4\int_{-1}^{1} \sqrt{1-x^2} \; dx$$

$$= 4 \int_{\pi}^{0} \underbrace{\sqrt{1-\cos^2 u}}_{\sin u} \cdot (-\sin u)\,du$$

$$= 4 \int_0^{\pi} \underbrace{\sin^2 u}_{= \frac{1-\cos 2u}{2}} \; du = 4\left(\frac{1}{2}u - \frac{\sin 2u}{4}\right)\Big|_{u=0}^{\pi}$$

$$= 4 \cdot \frac{\pi}{2} = 2\pi \quad \#$$

§ Triple integrals

Let $f = f(x, y, z)$

Similar as double integrals, if $T \subseteq \mathbb{R}^3$,

one can define the triple integral

$$\iiint_T f(x,y,z)\,dx\,dy\,dz \qquad \circledast$$

by a limit of Riemann sums.

Geometrically, such an integral computes

a "4-dimensional volume."

Alternatively, if one consider $f(x,y,z)$

as a density function, then $\circledast$ computes

the weight of T.

In particular, if $f(x,y,z) \equiv 1$, then

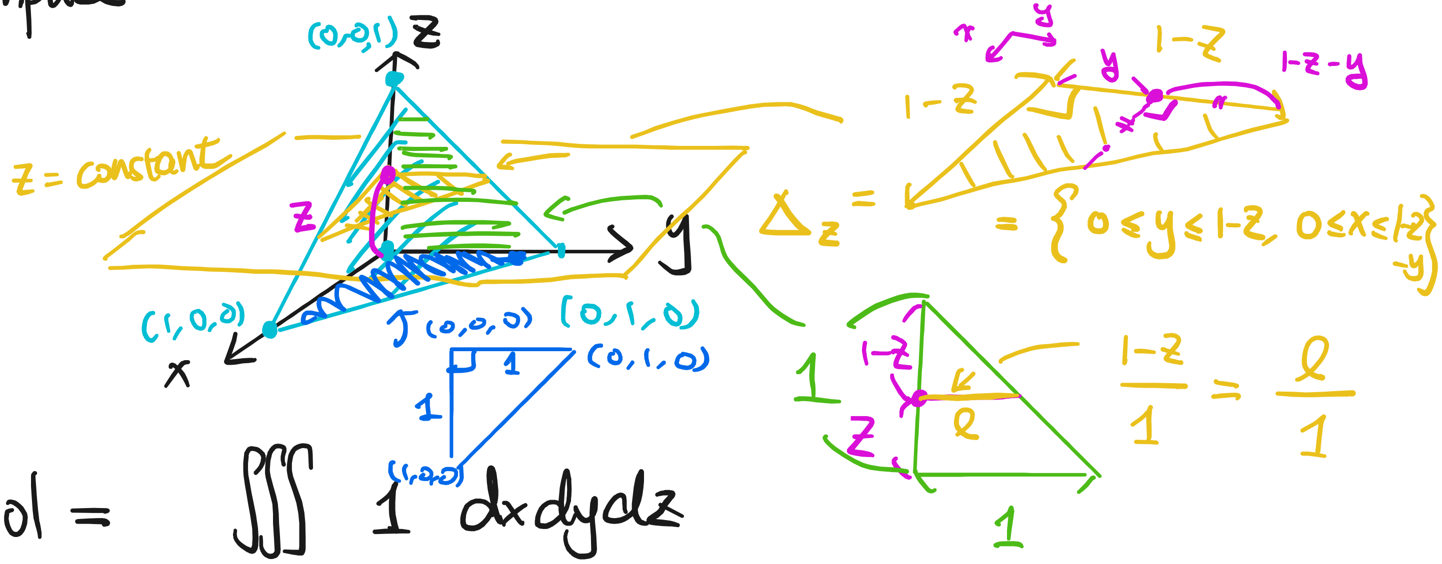
$$\iiint_T dx\,dy\,dz = \iiint_T 1\,dx\,dy\,dz$$

gives the volume of T.

Similar as double integrals, triple integrals can be computed by repeated integrals.

Example

Compute the volume of the tetrahedron T:



$$\text{vol} = \iiint_T 1 \, dx\,dy\,dz$$

$$= \int_0^1 \left(\iint_{\Delta_z} 1 \, dx\,dy \right) dz$$

$$= \int_0^1 \left(\int_0^{1-z} \left(\int_0^{1-z-y} 1 \, dx \right) dy \right) dz$$

$$= \int_0^1 \left(\int_0^{1-z} (1-z-y) \, dy \right) dz$$

$$= \int_0^1 \left. (1-z)y - \frac{y^2}{2} \right|_{y=0}^{1-z} dz$$

$$(1-z)^2 - \frac{(1-z)^2}{2} = \frac{(1-z)^2}{2} = \left(-\frac{(1-z)^3}{6}\right)'$$

$$= -\left. \frac{(1-z)^3}{6} \right|_{z=0}^{1} = \frac{1}{6} = \frac{1}{2} \times 1 \times \frac{1}{3}$$

#

Example

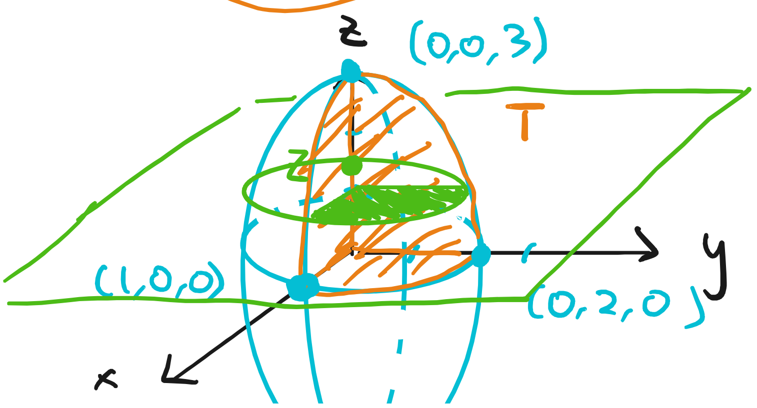
Integrate

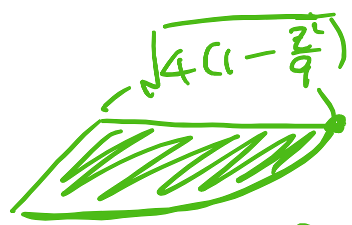
$$f(x,y,z) = xy$$

over the first octant solid T bounded by the coordinate planes and the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 \quad \Leftrightarrow \quad x^2 + \frac{y^2}{4} = \left(1 - \frac{z^2}{9}\right)$$

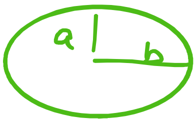
Sol





$$\frac{x^2}{1-\frac{z^2}{9}} + \frac{y^2}{4(1-\frac{z^2}{9})} = 1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$T=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; 0\le z\le 3,\ 0\le y\le \phi(z),\ 0\le x\le \psi(y,z)\right\}$$

where

$$\phi(z)=\sqrt{4\left(1-\frac{z^2}{9}\right)}$$

$$x^2=1-\frac{y^2}{4}-\frac{z^2}{9}$$

$$\psi(y,z)=\sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}}$$

So

$$\iiint_T xy\,dx\,dy\,dz=\int_0^3\left(\int_0^{2\sqrt{1-\frac{z^2}{9}}}\left(\underbrace{\int_0^{\sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}}} xy\,dx}_{}\right)dy\right)dz$$

$$\frac{x^2}{2}y\Big|_{x=0}^{\sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}}} = \frac{y}{2}\left(1-\frac{y^2}{4}-\frac{z^2}{9}\right)$$

$$\left(1-\frac{z^2}{9}\right)\frac{y^2}{4}-\frac{y^4}{32}\Big|_{y=0}^{2\sqrt{1-\frac{z^2}{9}}} = \left(1-\frac{z^2}{9}\right)^2-\frac{1}{2}\left(1-\frac{z^2}{9}\right)^2$$

$$=\int_0^3\left(\underbrace{\int_0^{2\sqrt{1-\frac{z^2}{9}}}\left(1-\frac{z^2}{9}\right)\frac{y}{2}-\frac{y^3}{8}\,dy}_{}\right)dz$$

$$=\int^3 \frac{1}{2}\left(1-\frac{z^2}{9}\right)^2 dz$$

$$= \frac{1}{2}\int_0^3 1 - \frac{2}{9}z^2 + \frac{1}{81}z^4 \, dz$$

$$= \frac{1}{2}\left(z - \frac{2}{27}z^3 + \frac{1}{81\times 5}z^5 \right)\Bigg|_{z=0}^{3}$$

$$= \frac{1}{2}\left(3 - 2 + \frac{3}{5}\right) = \frac{8}{10} = \frac{4}{5} \quad \#$$

Changing variables and Jacobians

Recall:

$$\int_a^b f(u(x)) \cdot u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du$$

For the case of functions of 2 variables:

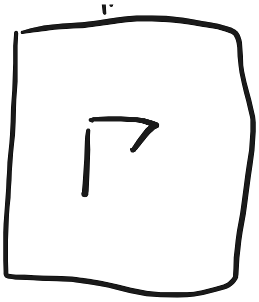
Thm (17.10.2)

Suppose

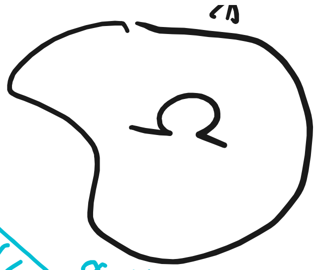
$$x = x(u,v) \qquad \text{and} \qquad y = y(u,v)$$

are smooth functions that map Γ to Ω

$$(u,v) \longrightarrow (x,y)$$

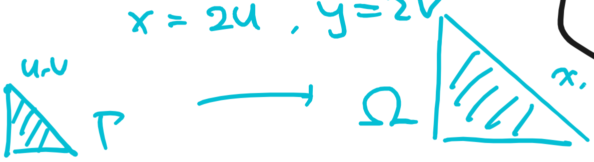


1-1, onto



e.g.

$x = 2u$, $y = 2v$



bijetively, and the Jacobian

$$J(u,v) := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

is nonzero on Γ. Then

e.g. $f = x^2 y$, $x=2u$, $y=2v$ $\quad f = 4u^2 \cdot 2v = 8u^2 v$

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Gamma} \tilde{f}(u,v)\,|J(u,v)|\,du\,dv$$

where

$$\tilde{f}(u,v) = f(x(u,v),\, y(u,v))$$

Remark

Geometrically, one can consider

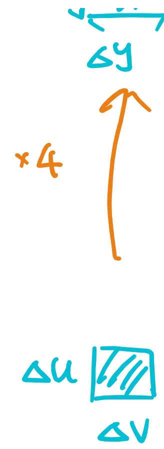
$\Delta x \cdot \Delta y$

$dx \rightsquigarrow$ small Δx $\quad$ Δx  $\quad$ $4\,\Delta u\,\Delta v$

$dy \rightsquigarrow$ small Δy

$du \rightsquigarrow$ small Δu

$dv \rightsquigarrow$ small Δv



$x = 2u$

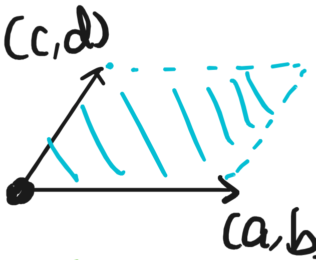
$y = 2v$

$\Downarrow$

$\Delta x = 2\Delta u$

$\Delta y = 2\Delta v$

Recall:



$$\text{area} = \left| \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right|$$

$\vec{\Delta x} = \left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right)$

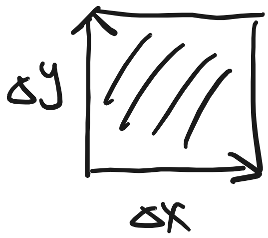
$\vec{\Delta y} = \left(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}\right)$

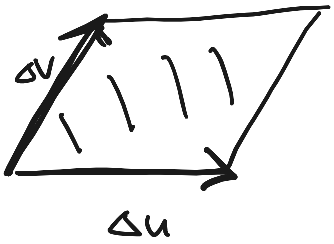
$(0,1) = \Delta v$

$\Delta u = (1,0)$

$$\vec{\Delta x} \approx \frac{\partial x}{\partial u} \cdot \vec{\Delta u} + \frac{\partial x}{\partial v} \cdot \vec{\Delta v}$$

$$\vec{\Delta y} \approx \frac{\partial y}{\partial u} \cdot \vec{\Delta u} + \frac{\partial y}{\partial v} \cdot \vec{\Delta v}$$



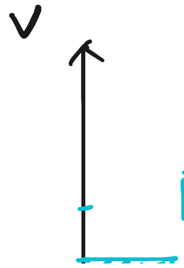


$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Example

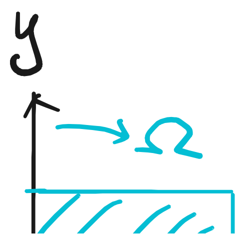
① $\displaystyle\iint_\Omega 1 \, dx\, dy$, $\Omega = \{0 \le x \le 2, 0 \le y \le 1\}$

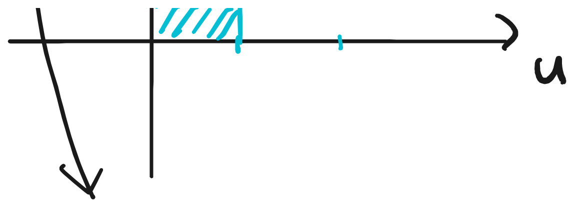
$x = 2u$, $y = 2v$

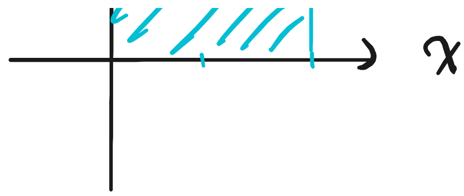


$\Gamma = \{0 \le u \le 1, 0 \le v \le \frac{1}{2}\}$

$\xrightarrow[y=2v]{x=2u}$







$$= \iint_{\Gamma} 1 \; |J(u,v)| \; du\,dv$$

where $J(u,v) = \det\begin{pmatrix} \overset{2}{\frac{\partial x}{\partial u}} & \overset{=0}{\frac{\partial y}{\partial u}} \\ \underset{=0}{\frac{\partial x}{\partial v}} & \underset{=2}{\frac{\partial y}{\partial v}} \end{pmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$$\iint_{\Omega} 1 \, dx\,dy = \iint_{\Gamma} 1 \cdot 4 \cdot du\,dv$$

$$\begin{array}{ccc} \iint_{\Omega} 1 \, dx\,dy & = & \iint_{\Gamma} 1 \cdot 4 \cdot du\,dv \\ \| & & \| \\ \text{area}(\Omega) & & 4 \ \text{area}(\Gamma) \\ \| & & \| \\ 2 \times 1 & = & 4\left(1 \times \frac{1}{2}\right) \end{array}$$

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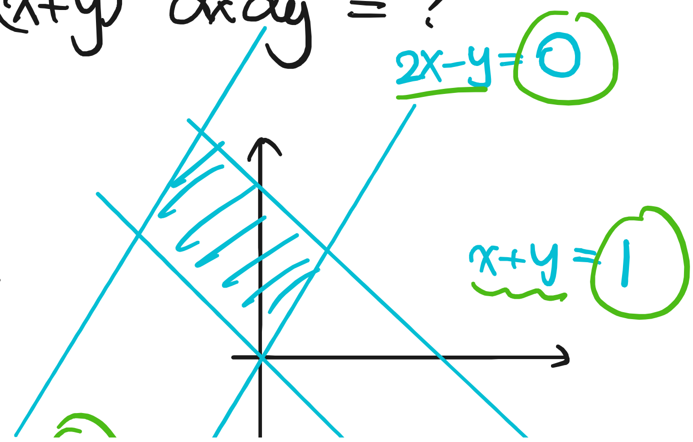
② $\iint_{\Omega} (x+y)^2 \, dx\,dy = ?$

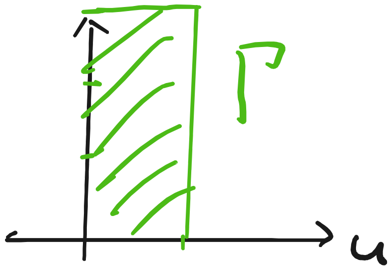
where

$\Omega =$



v

$2x-y=3$ $\quad$ $x+y=0$



Sol

Let

$$u = x+y, \quad v = 2x-y$$

$\Longleftrightarrow$

$$x = \frac{u+v}{3}, \quad y = \frac{2u-v}{3}$$

This transformation gives a bijection between Ω and Γ, where

$$\Gamma = \{ (u,v) \in \mathbb{R}^2 \mid 0 \le u \le 1, \ 0 \le v \le 3 \}$$

Since

$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

we have

$$\iint_{\Omega} (x+y)^2 \, dx\, dy$$

$$= \iint \Big(\overbrace{\frac{u+v}{3} + \frac{2u-v}{3}}^{u} \Big)^2 \cdot \left| -\frac{1}{3} \right| du\, dv$$

$$= \int_0^3 \left(\int_0^1 \frac{u^2}{3} \, du \right) dv$$

$$= \int_0^3 \underbrace{\left. \frac{u^3}{9} \right|_{u=0}^{1}}_{=\frac{1}{9}} dv = \frac{1}{9} \cdot 3 = \frac{1}{3}$$

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