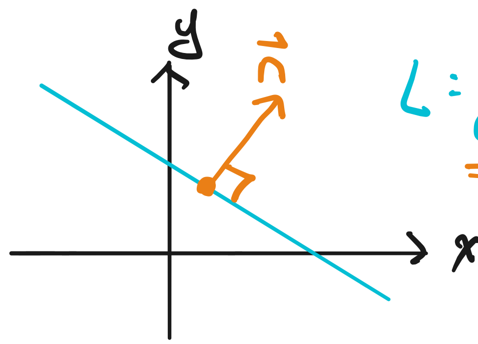
Calculus, week 11, Spring 2025

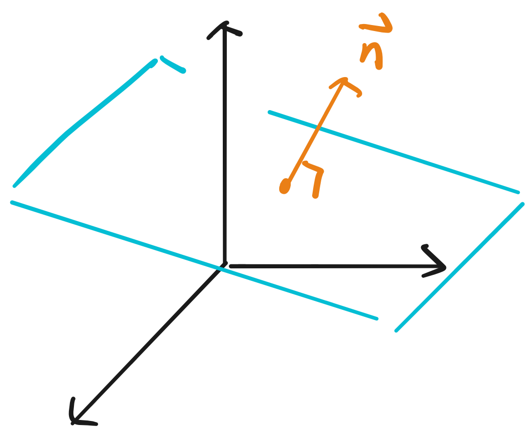
Recall

- Normal vectors of lines and planes



$L:\ \underline{a}x+\underline{b}y=c$

$\vec{n}=(a,b)$ is a normal vector of L



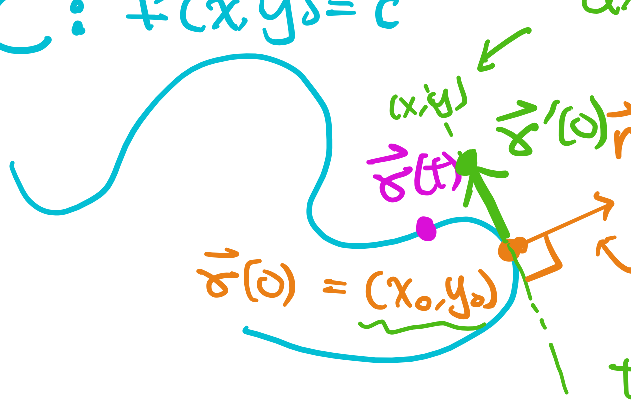
$P:\ \underline{a}x+\underline{b}y+\underline{c}z=d$

$\vec{n}=(a,b,c)$ is a normal vector of P

- Normal vectors of curves

$C:\ f(x,y)=c$

$ax+by=\underline{d} = \frac{\partial f}{\partial x}(x_0,y_0)\cdot x + \frac{\partial f}{\partial y}(x_0,y_0)\cdot y$



$\vec{n}$ = normal vector of C at (x_0,y_0)

$\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$

"$=(a,b)$"

tangent line

If C is parametrized by $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$ near $(x_0, y_0) = \vec{\gamma}(0)$ and $\vec{\gamma}\,'(0) \neq \vec{0}$, then

$$\vec{n} = \nabla f(x_0, y_0)$$

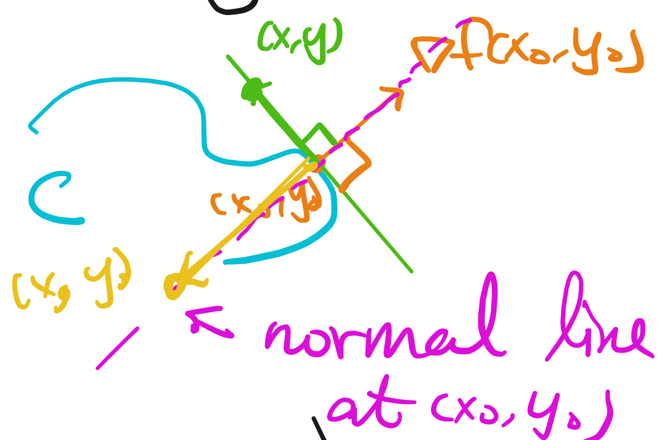
is a normal vector of C at (x_0, y_0)

Thm (16.4.4, 16.4.5)

Suppose $\nabla f(x_0, y_0) \neq 0$. The tangent line of the curve

$$C: f(x,y) = c$$

at (x_0, y_0) is



$$\nabla f(x_0,y_0) \bullet \big((x,y) - (x_0,y_0)\big) = 0$$

$$= \boxed{\frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0) = 0}$$

Furthermore, the normal line of C at (x_0, y_0), i.e. the normal line of tangent line at (x_0, y_0), is determined by $\left(x-x_0, y-y_0\right)$ $\left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$

$$\left((x,y)-(x_0,y_0)\right) \,//\, \nabla f(x_0,y_0)$$

i.e.

$$\boxed{\frac{\partial f}{\partial y}(x_0,y_0)\cdot(x-x_0) = \frac{\partial f}{\partial x}(x_0,y_0)\cdot(y-y_0)}$$

$r\vec{v}=\vec{w}$

$\vec{v}$

Remark

$\vec{v} \,//\, \vec{w} \iff \vec{v} = r\,\vec{w}$ for some $r \in \mathbb{R}$

or $\vec{w} = r\,\vec{v}$

If $\vec{w}=\vec{0}$, $\vec{v}\neq\vec{0}$, then $r\cdot\vec{0}=\vec{0}\neq\vec{v}$

but $0\cdot\vec{v}=\vec{0}=\vec{w}$

Assume $\vec{v}=(v_1,v_2)$

$\vec{w}=(w_1,w_2)$

Then

$\vec{v}=r\vec{w} \iff v_1=rw_1,\quad v_2=rw_2$ ①

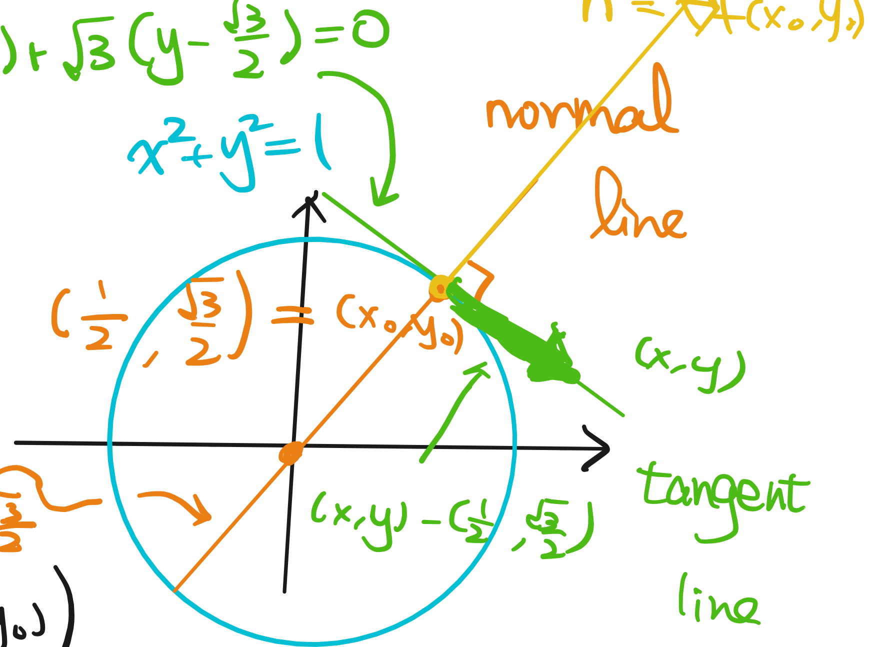
$\vec{w}=r\vec{v} \iff w_1=rv_1,\quad w_2=rv_2$ ②

$$\Rightarrow \quad v_1\cdot w_2 = \left\{\begin{array}{ll} ① & r w_1\cdot w_2 = w_1\cdot v_2 \\ ② & v_1\cdot r v_2 = w_1\cdot v_2 \end{array}\right\} = w_1\cdot v_2$$

Example

$$(x-\tfrac{1}{2})+\sqrt{3}\,(y-\tfrac{\sqrt{3}}{2})=0$$

$$C:\ \underbrace{x^2+y^2}_{=f(x,y)} = 1$$



$$\vec{n} = \nabla f(x_0,y_0)$$

$$\sqrt{3}\,(x-\tfrac{1}{2}) = y-\tfrac{\sqrt{3}}{2}$$

$$= \left(\frac{\partial f}{\partial x}(x_0,y_0),\ \frac{\partial f}{\partial y}(x_0,y_0)\right)$$

$= (2x_0,\ 2y_0)$ is normal vector of C at (x_0,y_0)

$= (1, \sqrt{3})$ at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$

The tangent line of C at $\underset{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}{\overset{\shortparallel}{(x_0,y_0)}}$ is

$$\left((x,y)-\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right)\right)\cdot(1,\sqrt{3}) = 0$$

$$= (x-\tfrac{1}{2}) + \sqrt{3}\cdot(y-\tfrac{\sqrt{3}}{2}) = 0$$

The normal line of C at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is

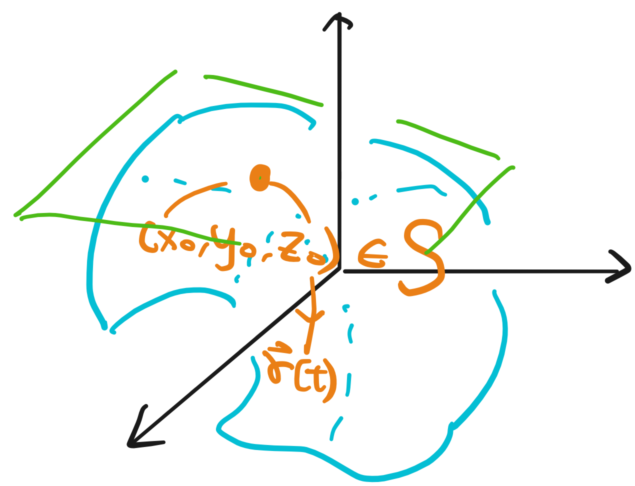
$$\left((x,y)-\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right)\right) \,/\!/\, (1,\sqrt{3})$$

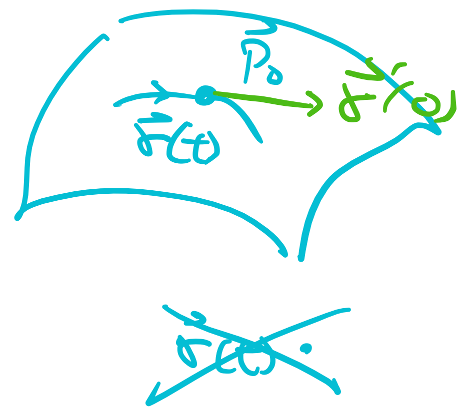
i.e.

$$\sqrt{3}\,(x-\tfrac{1}{2}) = 1\cdot(y-\tfrac{\sqrt{3}}{2})$$

Surface case: Consider

$$S: \; f(x,y,z) = c$$

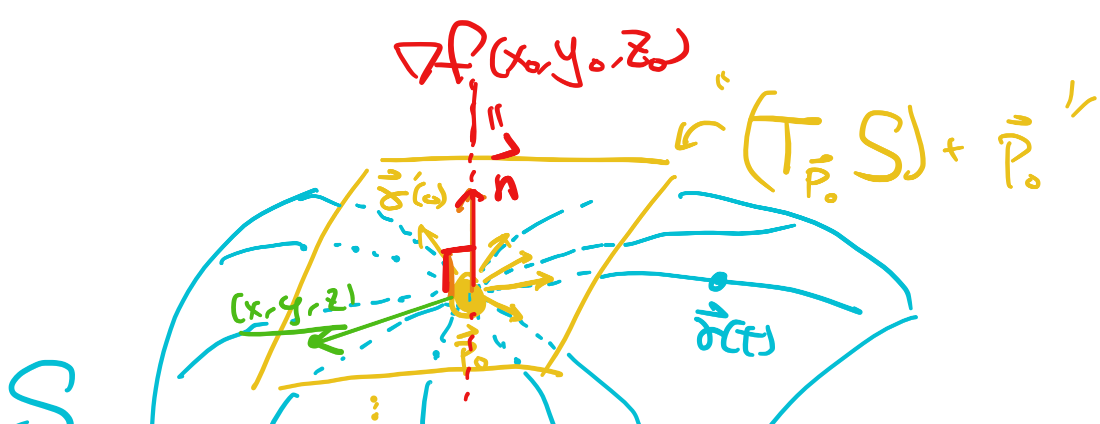




Assume $\vec{P}_0 = (x_0, y_0, z_0) \in S$, $\nabla f(x_0, y_0, z_0) \neq \vec{0}$

The tangent plane of S at (x_0, y_0, z_0) is

$$T_{\vec{P}_0} S = \left\{ \overset{(x,y,z)-(x_0,y_0,z_0)}{\overset{\shortparallel}{\vec{\gamma}'(0)}} \;\middle|\; \begin{array}{l} \vec{\gamma}: \mathbb{R} \to S \text{ smooth} \\ \vec{\gamma}(0) = \vec{P}_0 = (x_0, y_0, z_0) \end{array} \right\}$$



Given any curve $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$

in S with $\vec{\gamma}(0) = \vec{P}_0$, we have

$$f(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = C$$

$\Rightarrow$

$$\left.\frac{d}{dt}\right|_{t=0} f(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = 0$$

$$\|$$

$$\nabla f(\vec{\gamma}(0)) \bullet \vec{\gamma}\,'(0)$$

$$\|$$

$$\nabla f(x_0, y_0, z_0) \bullet \vec{\gamma}\,'(0)$$

<u>Thm</u> (16.4.8, 16.4.10)

Suppose $\nabla f(x_0, y_0, z_0) \neq \vec{0}$. The tangent plane of

$$S: f(x, y, z) = C$$

at (x_0, y_0, z_0) is determined by

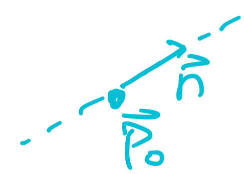
$$\nabla f(x_0, y_0, z_0) \bullet (x-x_0,\ y-y_0,\ z-z_0) = 0$$

$$= \boxed{\frac{\partial f}{\partial x}(x_0,y_0,z_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)\,(y-y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)\,(z-z_0) = 0}$$

Furthermore, the normal line (i.e. the normal line of the tangent plane) of S at (x_0, y_0, z_0) can be parametrized by

$$x = x_0 + t\cdot\frac{\partial f}{\partial x}(x_0,y_0,z_0) \qquad \vec{p} = \vec{p}_0 + t\cdot\vec{n}$$

$$y = y_0 + t\cdot\frac{\partial f}{\partial y}(x_0,y_0,z_0)$$



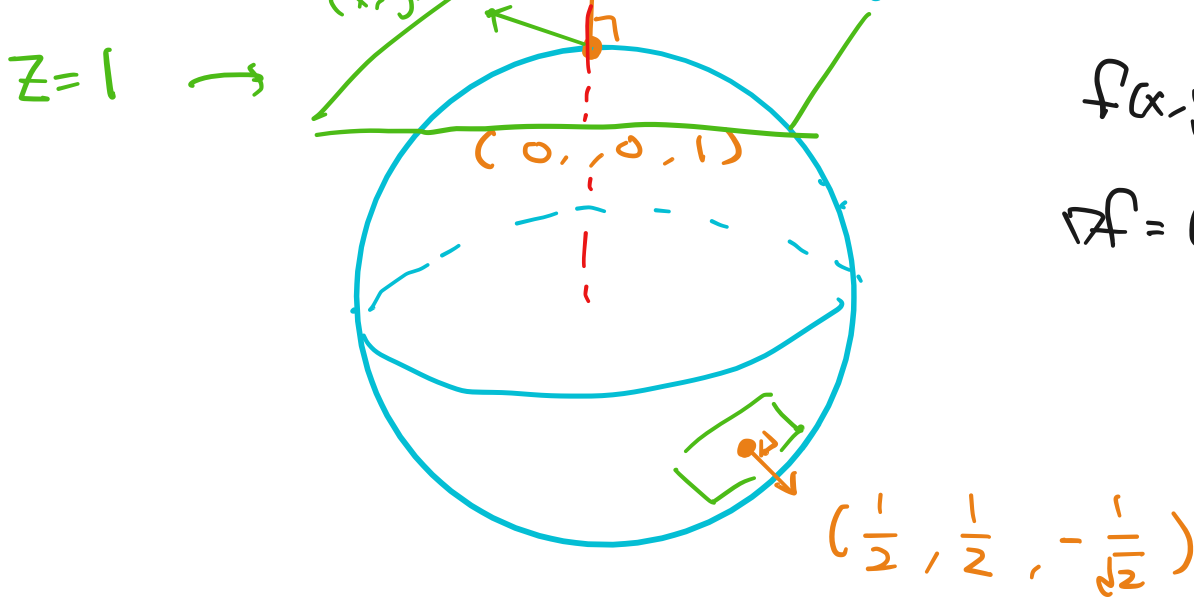
$$z = z_0 + t\,\frac{\partial f}{\partial z}(x_0,y_0,z_0)$$

Example

① Consider $S: \quad x^2+y^2+z^2=1$

$(0,0,1+2t)$ = z-axis

$x^2+y^2+z^2=1$



$f(x,y,z) = x^2+y^2+z^2$

$\nabla f = (2x, 2y, 2z)$

(i) $\vec{n} = \nabla f(0,0,1) = (0,0,2)$ is a normal vector of S at $(0,0,1)$

The tangent plane of S at $\cancel{(0,0,2)}$ $(0,0,1)$ is

$$(x-0,\ y-0,\ z-1) \cdot \underbrace{\nabla f(0,0,1)}_{(0,0,2)} = 0$$

$$\parallel$$

$$2(z-1) = 0$$

i.e. $z = 1$

The normal line of S at $(0,0,1)$ is

$$\begin{cases} x = 0 + t \cdot 0 = 0 \\ y = 0 + t \cdot 0 = 0 \\ z = 1 + t \cdot 2 = 1+2t \end{cases}$$

a parametrization of z-axis

(ii) At $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$,

$\vec{n} = \nabla f\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right) = (1, 1, -\sqrt{2})$

is a normal vector of S at $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

The tangent plane of S at $\left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$ is

$$(1, 1, -\sqrt{2}) \bullet \left(x - \frac{1}{2}, y - \frac{1}{2}, z + \frac{1}{\sqrt{2}}\right) = 0$$

i.e.

$$\left(x - \frac{1}{2}\right) + \left(y - \frac{1}{2}\right) - \sqrt{2}\left(z + \frac{1}{\sqrt{2}}\right) = 0$$

The normal line of S at $\left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$ is

$$\begin{cases} x = \frac{1}{2} + t \cdot 1 \\ y = \frac{1}{2} + t \cdot 1 \\ y = -\frac{1}{\sqrt{2}} + t(-\sqrt{2}) \end{cases}$$

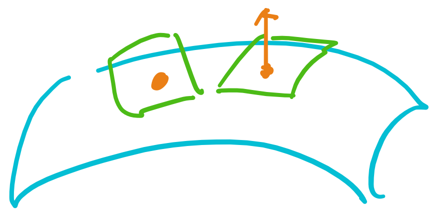
#

② At what point(s) of the surface

$z = 3xy - x^3 - y^3$ $\Leftrightarrow$ $\underbrace{z + x^3 + y^3 - 3xy}_{f(x,y,z)} = 0$

is the tangent plane horizontal ?

水平的 i.e. " $z = c$ "



<u>sol</u>

∇f

Observation:

A plane is horizontal $\Leftrightarrow$ its normal vector is parallel to $(0,0,1)$

So we shall solve

arbitrary

$$\nabla f = (0, 0, \underline{\underline{r}})$$

i.e. slove $f = z + x^3 + y^3 - 3xy$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 = 3x^2 - 3y = 3(x^2 - y) \\ \dfrac{\partial f}{\partial y} = 0 = 3y^2 - 3x = 3(y^2 - x) \end{cases}$$

$$\Rightarrow \begin{cases} x^2 = y \\ y^2 = x \end{cases} \Rightarrow x^4 = y^2 = x$$

$$\Rightarrow \underbrace{x^4 - x}_{\shortparallel} = 0$$

$x(x^3-1)$

$\Rightarrow \quad x = 0, 1$

$(y = 0 \quad 1)$

So the tangent planes at

$(0, 0, 0)$

and

$(1, 1, 1)$

are horizontal. #