Calculus, week 10, Spring 2025

<u>Recall</u>

① If $f: \mathbb{R}^3 \to \mathbb{R}$, then the gradient of f at $\vec{p} \in \mathbb{R}^3$ is

$$\nabla f(\vec{p}) = \frac{\partial f}{\partial x}(\vec{p})\,\vec{i} + \frac{\partial f}{\partial y}(\vec{p})\,\vec{j} + \frac{\partial f}{\partial z}(\vec{p})\,\vec{k}$$

$$= \left(\frac{\partial f}{\partial x}(\vec{p}), \frac{\partial f}{\partial y}(\vec{p}), \frac{\partial f}{\partial z}(\vec{p}) \right)$$

② The directional derivative of f along a unit vector $\vec{u}$ at $\vec{p}$ is

$$(\nabla_{\vec{u}} f)(\vec{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h}$$

$$\overset{\text{Thm}}{=} \nabla f(\vec{p}) \cdot \vec{u}$$

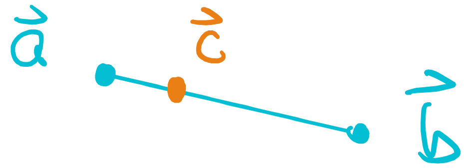
§ Mean value thm and chain rule

<u>Thm</u> (Thm 16.3.1)

Let f be a smooth function of n variables. Given $\vec{a}, \vec{b} \in \mathbb{R}^n$, $\exists$ a point $\vec{c}$ on the line segment connecting $\vec{a}$ and $\vec{b}$

s.t.

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a}) \quad (*)$$



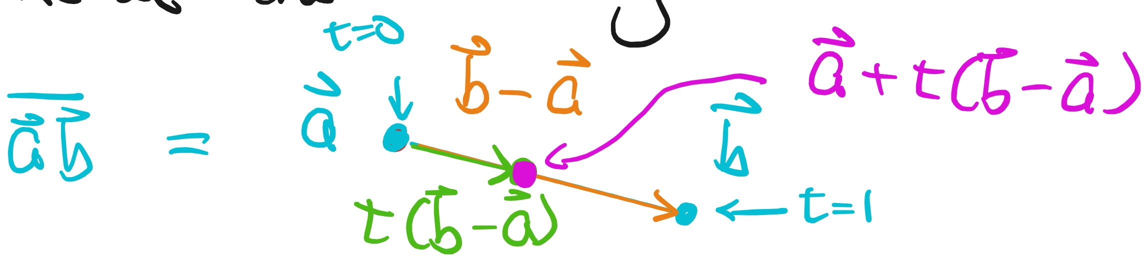
pf

If $\vec{a} = \vec{b}$, then the thm is true

(LHS of $(*)$ = RHS of $(*)$ = 0)

Assume $\vec{a} \neq \vec{b}$.

Recall the line segment



can be parametrized by

$$\underbrace{\vec{a} + t(\vec{b} - \vec{a})}_{= \vec{a} + t \cdot \|\vec{b}-\vec{a}\| \cdot \vec{u}}, \qquad t \in [0, 1].$$

Let $\vec{u} = \dfrac{\vec{b} - \vec{a}}{\|\vec{b} - \vec{a}\|}$

Consider the restriction of f to $\overline{\vec{a}\vec{b}}$:

$$g(t) = f(\vec{a} + t(\vec{b} - \vec{a})), \qquad t \in [0, 1]$$

$$g(t) = f(\vec{a} + t(\vec{b}-\vec{a})),\quad t \in [0,1]$$

$$= f(\vec{a} + \underline{t\|\vec{b}-\vec{a}\|}\,\vec{u})$$

NOTE:

$$\frac{d}{ds} f(\vec{a} + s\vec{u}) = \lim_{h\to 0} \frac{f(\overbrace{\boxed{\vec{a}+s\vec{u}}}^{\vec{p}} + h\vec{u}) - f(\overbrace{\boxed{\vec{a}+s\vec{u}}}^{\vec{p}})}{h}$$

$$= \nabla_{\vec{u}} f(\vec{a}+s\vec{u}) \overset{\text{Thm}}{=} \nabla f(\vec{a}+s\vec{u}) \bullet \vec{u}$$

Therefore $\left(s = t\cdot\|\vec{b}-\vec{a}\| \Rightarrow g(t) = f(\vec{a}+s\vec{u}) \right)$

$$g'(t) = \frac{dg}{ds}\ \underbrace{\frac{ds}{dt}}_{=\|\vec{b}-\vec{a}\|}$$

$$= \left(\nabla f(\vec{a} + \underbrace{t\cdot\|\vec{b}-\vec{a}\|\cdot\vec{u}}_{=\vec{b}-\vec{a}}) \bullet \underbrace{\vec{u}}_{=\frac{\vec{b}-\vec{a}}{\cancel{\|\vec{b}-\vec{a}\|}}} \right) \cancel{\|\vec{b}-\vec{a}\|}$$

$$= \nabla f(\vec{a} + t(\vec{b}-\vec{a})) \bullet (\vec{b}-\vec{a})$$

By the mean value thm (apply to $g(t)$),

$\exists\ t_0 \in (0,1)$ s.t.

$$g(1) - g(0) = g'(t_0)\cdot(1-0)$$

$$\| \qquad\qquad\qquad \|$$

$$f(\vec{b}) - f(\vec{a}) \qquad \nabla f(\vec{a} + t_0(\vec{b}-\vec{a})) \bullet (\vec{b}-\vec{a})$$

$f(\vec b) - f(\vec a) = \nabla f(\vec a + t_0(\vec b - \vec a)) \cdot (\vec b - \vec a)$

Let $\vec c = \vec a + t_0(\vec b - \vec a)$. Then

$$f(\vec b) - f(\vec a) = \nabla f(\vec c) \cdot (\vec b - \vec a) \quad \#$$

<u>Thm</u>

(i) If $\nabla f(\vec x) = \vec 0 \quad \forall\, \vec x \in \mathbb{R}^3$, then

f is constant. i.e. $f(\vec x) = f(\vec y) \quad \forall\, \vec x, \vec y \in \mathbb{R}^3$.

(ii) If $\nabla f(\vec x) = \nabla g(\vec x) \quad \forall\, \vec x \in \mathbb{R}^3$, then

$\exists\, c \in \mathbb{R}$ s.t.

$$f(\vec x) = g(\vec x) + c \qquad \forall\, \vec x \in \mathbb{R}^3$$

pf

(i) Given any $\vec x, \vec y \in \mathbb{R}^3$, by MVT, $\exists\, \vec c \in \overline{\vec x \vec y}$

s.t.

$$f(\vec x) - f(\vec y) = \overset{\vec 0}{\overset{\parallel}{\nabla f(\vec c)}} \cdot (\vec x - \vec y)$$

$$= 0$$

$$\Rightarrow \quad f(\vec x) = f(\vec y) \qquad \forall\, \vec x, \vec y \in \mathbb{R}^3.$$

$$= f(\vec 0)$$

(ii) Consider $f-g$

$$\nabla(f-g)(\vec{x}) = \nabla f(\vec{x}) - \nabla g(\vec{x}) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^3.$$

$$\nabla(f-g)(\vec{x}) = ((f-g)_x, (f-g)_y, (f-g)_z) = (f_x - g_x, f_y - g_y, f_z - g_z) = (f_x, f_y, f_z) - (g_x, g_y, g_z)$$

By (i), $f(\vec{x}) - g(\vec{x}) = \text{constant} = C \in \mathbb{R}$ $\quad \forall \vec{x} \in \mathbb{R}^3$

$\Rightarrow \quad f(\vec{x}) = g(\vec{x}) + C \quad \forall \vec{x} \in \mathbb{R}^3$

#

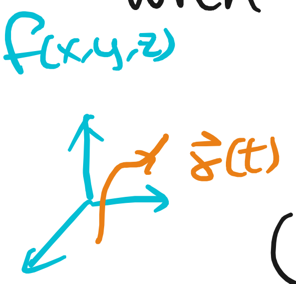
<u>Chain rule</u>

smooth

$f = f(x,y,z)$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\vec{\gamma}"] \arrow[rr, bend left] & \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Let $\vec{\gamma}(t)$ be vector-valued differentiable function with one variable. Consider



$$f \circ \vec{\gamma} : \mathbb{R} \to \mathbb{R}$$

$$(f \circ \vec{\gamma})(t) = f(\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

Thm (Chain rule Thm 16.3.4) $\cap$ $\quad f(\vec{\gamma}(t+h)) - f(\vec{\gamma}(t))$

Thm (Chain Rule, Thm ...)

$$\lim_{h\to 0} \frac{\quad}{h}$$

$$(f\circ\vec{\gamma})'(t) = \nabla f(\vec{\gamma}(t)) \bullet \vec{\gamma}\,'(t)$$

$$\| $$

$$\frac{d}{dt}\Big(f(\gamma_1(t), \gamma_2(t), \gamma_3(t))\Big)$$

$$(\gamma_1'(t), \gamma_2'(t), \gamma_3'(t))$$

$$\|$$

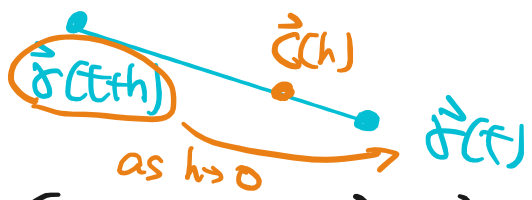
$$\left(\frac{\partial f}{\partial x}(\vec{\gamma}(t)), \frac{\partial f}{\partial y}(\vec{\gamma}(t)), \frac{\partial f}{\partial z}(\vec{\gamma}(t))\right) \bullet \vec{\gamma}\,'(t)$$

$$\begin{aligned} = \; & \frac{\partial f}{\partial x}(\gamma_1(t), \gamma_2(t), \gamma_3(t)) \cdot \gamma_1'(t) \\ + \; & \frac{\partial f}{\partial y}(\gamma_1(t), \gamma_2(t), \gamma_3(t)) \cdot \gamma_2'(t) \\ + \; & \frac{\partial f}{\partial z}(\gamma_1(t), \gamma_2(t), \gamma_3(t)) \cdot \gamma_3'(t) \end{aligned}$$

pf

By MVT, $\exists\ \vec{c} = \vec{c}(h)$ between $\vec{\gamma}(t+h)$ and $\vec{\gamma}(t)$

s.t.



$$f(\vec{\gamma}(t+h)) - f(\vec{\gamma}(t)) = \nabla f(\vec{c}(h)) \bullet (\vec{\gamma}(t+h) - \vec{\gamma}(t))$$

$\Rightarrow$

$$\frac{f(\vec{\gamma}(t+h)) - f(\vec{\gamma}(t))}{h}$$

$$= \frac{1}{h}\left(\nabla f(\vec{c}(h)) \bullet (\vec{\gamma}(t+h) - \vec{\gamma}(t))\right)$$

$$= \nabla f(\boxed{\vec{c}(h)}) \bullet \boxed{\left(\frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}\right)}$$

$\downarrow$ as $h \to 0$ $\quad\quad$ $\downarrow$ as $h \to 0$

$\vec{\gamma}(t)$ $\quad\quad$ $\vec{\gamma}'(t)$

Recall

$$\vec{\gamma}'(t) = \lim_{h\to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

Since f is smooth, f_x, f_y, f_z are continuous

$$\Rightarrow \lim_{h\to 0} f_x(\vec{c}(h)) = f_x\left(\lim_{h\to 0} \vec{c}(h)\right) = f_x(\vec{\gamma}(t))$$

$$\lim_{h\to 0} f_y(\vec{c}(h)) = f_y(\vec{\gamma}(t)), \quad \lim_{h\to 0} f_z(\vec{c}(h)) = f_z(\vec{\gamma}(t))$$

$$\Rightarrow \lim_{h\to 0} \nabla f(\vec{c}(h)) = (f_x(\vec{\gamma}(t)), f_y(\vec{\gamma}(t)), f_z(\vec{\gamma}(t)))$$

$$= \nabla f(\vec{\gamma}(t))$$

S

so

$$(f\circ\vec{r})'(t) = \lim_{h\to 0} \frac{f(\vec{r}(t+h)) - f(\vec{r}(t))}{h}$$

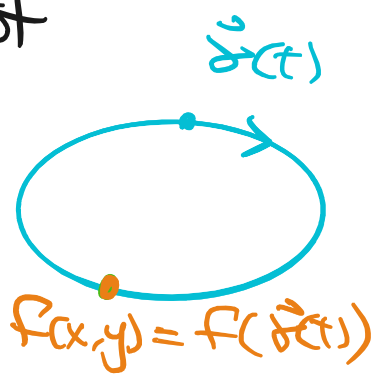
$$= \lim_{h\to 0}\left(\underbrace{\nabla f(\vec{c}(h))}_{\nabla f(\vec{r}(t))} \bullet \underbrace{\frac{\vec{r}(t+h) - \vec{r}(t)}{h}}_{\vec{r}\,'(t)} \right)$$

$$= \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t) \qquad \#$$

Example

① Find the rate of change of

$$f(x,y) = \frac{1}{3}(x^3 + y^3)$$



along the curve

$$\vec{r}(t) = (a\cos t,\ b\sin t)$$

with respect to t.

sol

$$\frac{d}{dt} f(\vec{r}(t)) \overset{\text{chain}}{\underset{\text{rule}}{=}} \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t)$$

$$\nabla f = (f_x, f_y) = (x^2, y^2)$$

$$\nabla f(\vec{\gamma}(t)) = ((a\cos t)^2, (b\sin t)^2)$$

$$= (a^2\cos^2 t, \quad b^2 \sin^2 t)$$

$$\vec{\gamma}'(t) = (-a\sin t, \quad b\cos t)$$

$$\Rightarrow \text{ans} = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$= (a^2\cos^2 t)(-a\sin t) + (b^2\sin^2 t)(b\cos t)$$

$$= \sin t\cos t\,(b^3 \sin t - a^3 \cos t) \quad \#$$

② Assume

$$u = x^2 - y^2$$

$$x = t^2 - 1$$

$$y = 3$$

Q: $\frac{du}{dt} = ?$

sol

$$\frac{du}{dt} \overset{\text{chain}}{=} \overset{2x = 2(t^2-1)}{\frac{\partial u}{\partial x}} \overset{2t}{\frac{dx}{dt}} + \overset{-2y}{\frac{\partial u}{\partial y}} \overset{=0}{\frac{dy}{dt}}$$

$$\frac{du}{dt} \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\| $$

$$\nabla u \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$$= 2(t^2-1)\cdot 2t + 0$$

$$= 4t^3 - 4t \qquad \#$$

③ Assume

$$u = x^2 - 2xy + 2y^3$$

$$x = s^2 \ln t, \qquad y = 2st^3$$

Q: $\frac{\partial u}{\partial s} = ?$ $\qquad \frac{\partial u}{\partial t} = ?$

sol

$$\frac{\partial u}{\partial s} \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x-2y)(2s\ln t) + (-2x+6y^2)(2t^3)$$

$$= (2s^2\ln t - 4st^3)\cdot 2s\ln t$$

$$+ (-2s^2\ln t + 24s^2t^6)\cdot 2t^3$$

$$\frac{\partial u}{\partial t} \overset{\text{chain}}{=} \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} \underset{\text{rule}}{=} \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2s^2 \ln t - 4st^3)\cdot \frac{s^2}{t}$$

$$+ (-2s^2 \ln t + 24 s^3 t^6)\cdot 6st^2 \quad \#$$

④ Assume

$y = y(x)$ is differentiable

and

$$u(x,y) = 2x^2y - y^3 + 1 - x - 2y = 2$$

Q: $\frac{dy}{dx} = ?$

<u>sol</u>

$$0 = \frac{d(u(x,y))}{dx} \underset{\text{rule}}{\overset{\text{chain}}{=}} u_x \cdot \frac{dx}{dx} + u_y \frac{dy}{dx}$$

$$= (4xy - 1)\cdot 1 + (2x^2 - 3y^2 - 2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - 4xy}{2x^2 - 3y^2 - 2} \quad \#$$

D... $f(x,y,z)$ $\vec{r}(t)$

Remark $f(x,y,z)$ $\vec{\gamma}(t)$

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R} \xrightarrow{\vec{\gamma}} \mathbb{R}^3$$

$$\vec{\gamma}(f(x,y,z)) = (\gamma_1(f(x,y,z)), \gamma_2(f(x,y,z)), \gamma_3(f(x,y,z)))$$

$$\frac{\partial(\gamma_1 \circ f)}{\partial x} = \gamma_1'(f(x,y,z)) \cdot \frac{\partial f}{\partial x}(x,y,z)$$

$$\begin{pmatrix} \frac{\partial(\gamma_1\circ f)}{\partial x} & \frac{\partial(\gamma_1\circ f)}{\partial y} & \frac{\partial(\gamma_1\circ f)}{\partial z} \\ \frac{\partial(\gamma_2\circ f)}{\partial x} & \cdots & \\ & \cdots & \end{pmatrix}_{3\times 3}$$

$$= \begin{pmatrix} \gamma_1' \\ \gamma_2' \\ \gamma_3' \end{pmatrix}_{3\times 1} \begin{pmatrix} f_x & f_y & f_z \end{pmatrix}_{1\times 3}$$

Recall

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\vec{\gamma}"] \arrow[rr, bend right, "f\circ\vec{\gamma}"'] & \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$(f\circ\vec{\gamma})'(t) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

Or

$$u = f(x,y,z)$$

$\gamma_1(t)$ $\gamma_2(t)$ $\gamma_3(t)$

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

$$u(t) = f(x(t), y(t), z(t)) \qquad \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial y}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Example

Assume $z = z(x, y)$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

① $z^4 + x^2 z^3 + y^2 + xy = 2$

Sol

$$\frac{\partial ①}{\partial x}: \quad \frac{\partial}{\partial x}(\underbrace{z^4 + x^2 z^3 + y^2 + xy}_{u(x,y,z)}) = \frac{\partial}{\partial x}(2) = 0$$

$$= \frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial u}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= (2xz^3 + y) \cdot 1 + (4z^3 + 3x^2 z^2) \cdot \frac{\partial z}{\partial x}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-(2xz^3 + y)}{4z^3 + 3x^2 z^2}$$

$$\frac{\partial ①}{\partial y}: \quad \frac{\partial}{\partial y}(z^4 + x^2 z^3 + y^2 + xy) = \frac{\partial}{\partial y}(2) = 0$$

$$\shortparallel$$

$$\frac{\partial u}{\partial x}\cdot\overset{=0}{\frac{\partial x}{\partial y}}+\frac{\partial u}{\partial y}\cdot\overset{=1}{\frac{\partial y}{\partial y}}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}$$

$$=(2y+x)\cdot 1+(4z^3+3x^2z^2)\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}=\frac{-(2y+x)}{4z^3+3x^2z^2}$$

#

② $\underbrace{\cos(xyz)+\ln(x^2+y^2+z^2)}_{u(x,y,z)}=0$

Sol

$$\frac{\partial ②}{\partial x}:\quad \frac{\partial u}{\partial x}\cdot\overset{=1}{\frac{\partial x}{\partial x}}+\frac{\partial u}{\partial y}\cdot\overset{=0}{\frac{\partial y}{\partial x}}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}0=0$$

$$\left(-\sin(xyz)\cdot yz+\frac{2x}{x^2+y^2+z^2}\right)\cdot 1$$

$$+\left(-\sin(xyz)\,xy+\frac{2z}{x^2+y^2+z^2}\right)\cdot\frac{\partial z}{\partial x}$$

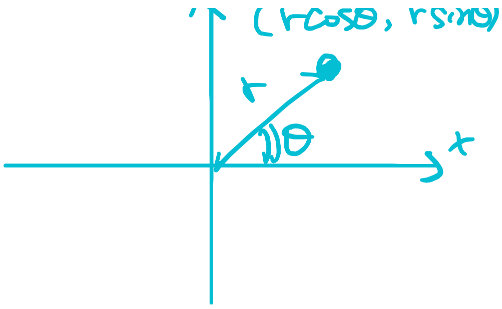
$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\sin(xyz)\cdot yz-\frac{2x}{x^2+y^2+z^2}}{-\sin(xyz)\,xy+\frac{2z}{x^2+y^2+z^2}}$$

$\frac{\partial z}{\partial y}$ can be computed similarly

#

y

Example (Polar coordinates)



Assume $u = u(x,y)$ is smooth

$= u(r,\theta)$

Suppose

$$x = r\cos\theta$$
$$y = r\sin\theta$$

① $\dfrac{\partial u}{\partial r} = ?$ $\qquad \dfrac{\partial u}{\partial \theta} = ?$

Sol

$$\frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$$

$$\frac{\partial u}{\partial r} \underset{\text{rule}}{\overset{\text{Chain}}{=}} \frac{\partial u}{\partial x}\cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$$

$$= \frac{\partial u}{\partial x}\cdot \cos\theta + \frac{\partial u}{\partial y}\cdot \sin\theta$$

$$\frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$$

$$= -\frac{\partial u}{\partial x}\cdot r\sin\theta + \frac{\partial u}{\partial y}\cdot r\cos\theta \qquad \#$$

② Assume $r \neq 0$.

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = ?$$

<u>sol</u>

By ①,

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta\right)^2$$

$$+ \frac{1}{\cancel{r^2}}\left(-\frac{\partial u}{\partial x}\cdot \cancel{r}\sin\theta + \frac{\partial u}{\partial y}\cdot \cancel{r}\cos\theta\right)^2$$

$$= \boxed{\left(\frac{\partial u}{\partial x}\right)^2}\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cdot\sin^2\theta + \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\sin\theta}$$

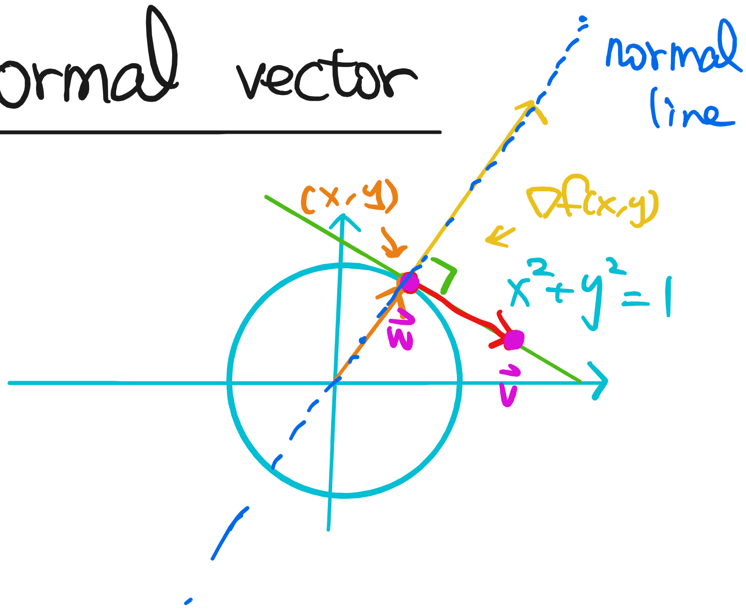
$$+ \boxed{\left(\frac{\partial u}{\partial x}\right)^2}\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cdot\cos^2\theta - \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta+\cos^2\theta)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \quad (= \Delta u)$$

#

§ Gradient and normal vector



Example

$$f(x,y) = x^2+y^2=1$$

$$\nabla f(x,y) = (2x, 2y)$$

$$= 2 \cdot (x,y)$$

Def

法向量

A normal vector $\vec{n}$ of a line L in $\mathbb{R}^2$ is a vector that is perpendicular to L, i.e.

$$\vec{n} \cdot (\vec{v} - \vec{w}) = 0$$

$\forall \ \vec{v}, \vec{w} \in L$

The normal line of L is the line that is perpendicular to L.

Assume a line L in $\mathbb{R}^2$ is given by

Assume a line L in

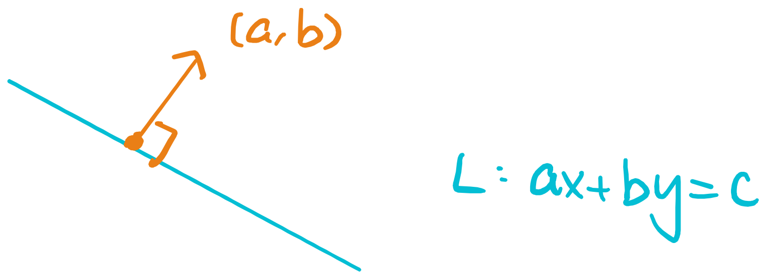
$$ax + by = c$$
$$\parallel$$
$$(a,b) \cdot (x,y)$$

Then for $\vec{v}, \vec{w} \in L$,

$$(a,b) \cdot \vec{v} = c$$
$$-)\quad (a,b) \cdot \vec{w} = c$$

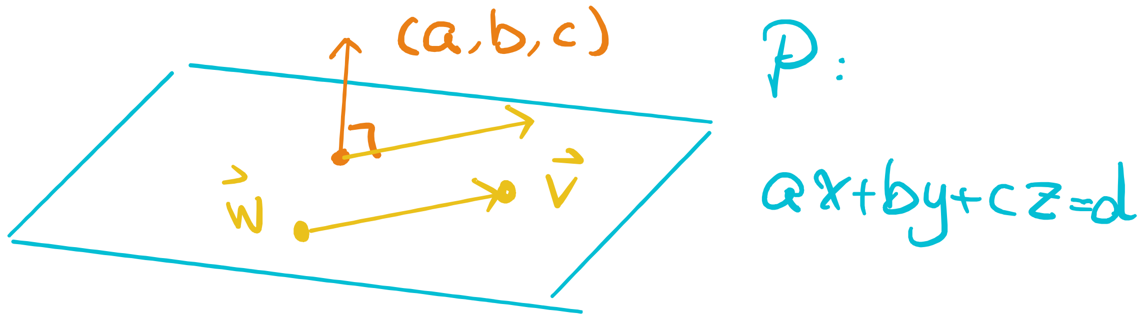
$$\Rightarrow \quad (a,b) \cdot (\vec{v} - \vec{w}) = c - c = 0$$

So (a,b) is a normal vector of L.



Similarly, a normal vector of a plane P in $\mathbb{R}^3$ is a vector that is perpendicular to P, and a normal line of P

is a line that is perpendicular to P.



(v_1, v_2, v_3)

If $\vec{v}, \vec{w} \in P$, then

(w_1, w_2, w_3)

$$\underline{a v_1 + b v_2 + c v_3} = c$$

$$= (a,b,c) \bullet \vec{v}$$

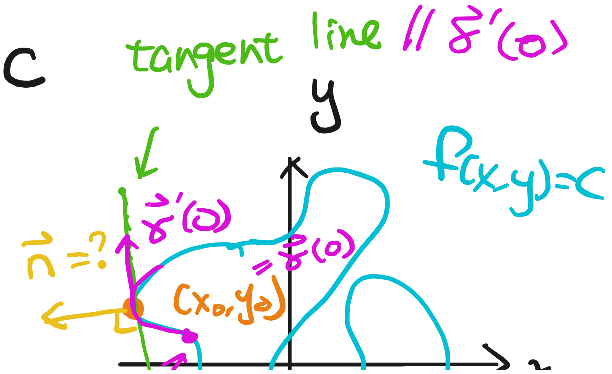
$$\underline{a w_1 + b w_2 + c w_3} = c$$

$$= (a,b,c) \bullet \vec{w}$$

$$\Rightarrow \quad (a,b,c) \bullet (\vec{v} - \vec{w}) = 0$$

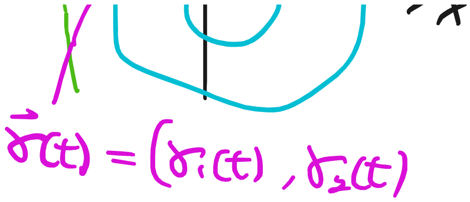
Consider a curve in xy-plane

$$C: \quad f(x,y) = c$$



Suppose $(x_0, y_0) \in C$ and

$\nabla f(x_0, y_0) \neq \vec{0}$



Suppose, near (x_0, y_0),

C is parametrized by a differentiable function

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t)), \quad \vec{\gamma}(0) = (x_0, y_0)$$

$\Rightarrow$

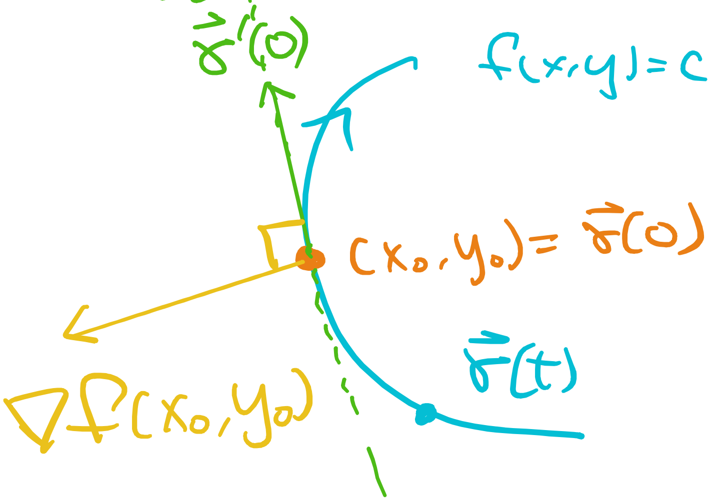
$$f(\vec{\gamma}(t)) = f(\gamma_1(t), \gamma_2(t)) = c$$

$$\Rightarrow \quad \frac{d}{dt}\left(f(\gamma_1(t), \gamma_2(t))\right) = \frac{d}{dt}(c) = 0$$

$$= \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

At $t=0$:

$$\nabla f(x_0, y_0) \cdot \underbrace{\vec{\gamma}'(0)}_{\vec{\gamma}'(0)} = 0$$



So $\nabla f(x_0, y_0)$ is a normal vector of the tangent line of C at (x_0, y_0)

Difference between

$\frac{\partial f}{\partial x}$ and $\frac{df}{dx}$

For example, if

$$f(x,y) = x^2 + y^2, \qquad y = y(x)$$

then

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{df}{dx} = \frac{d\, x^2 + (y(x))^2}{dx} = 2x + 2y \cdot \frac{dy}{dx}$$

HW6. 2:

Assume f is entire and

$$f''(x) + x f(x) = 0 \qquad - \circledast$$

Q: $f(x) = ?$

Sol

Since f is entire, $\exists\ a_k \quad k! \ )$ s.t.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \forall\, x \in (-\infty, \infty).$$

Thm
$\Rightarrow$
$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \qquad \forall x \in (-\infty, \infty)$$

$$\Rightarrow \quad f''(x) = \sum_{k=2}^{\infty} k(k-1)\, a_k\, x^{k-2} \qquad \forall x \in (-\infty, \infty)$$

$\circledast$:

$$f''(x) + x f(x) = 0$$

$$\sum_{\substack{k=2 \\ k=-1}}^{\infty} \overset{(k+3)(k+2)}{\overset{\parallel}{k(k-1)}}\, a_{\cancel{k}\ k+3}\; x^{\overset{k+1}{k-2}} + \cancel{x} \sum_{k=0}^{\infty} a_k \cdot x^{k+1}$$

$$= \sum_{k=-1}^{\infty} (k+3)(k+2)\, a_{k+3}\, x^{k+1} + \sum_{k=0}^{\infty} a_k \cdot x^{k+1}$$

$$= 2 \cdot 1 \cdot a_2 + \sum_{k=0}^{\infty} \Big((k+3)(k+2) a_{k+3} + a_k \Big) x^{k+1}$$

$$= 0$$

$$\Rightarrow \begin{cases} 2a_2 = 0 \Rightarrow a_2 = 0 \end{cases}$$

Remark

$$\sum_{k=0}^{\infty} b_k x^k = 0 \quad \forall x$$

$$\Rightarrow b_k = 0 \quad \forall k$$

$$\left((k+3)(k+2)\, a_{k+3} + a_k = 0 \right. \quad \forall k \geq 0$$

$$\Downarrow$$

$$a_{k+3} = - \frac{a_k}{(k+3)(k+2)}$$

$$\left(\because\ b_0 = (\quad)\big|_{x=0} = 0 \right.$$

$$b_1 = (\quad)'\big|_{x=0}$$

$$= 0'\big|_{x=0} = 0$$

$$2b_2 = (\quad)''\big|_{x=0} = 0$$

$$\vdots \qquad \Big)$$

$k=0$

$$a_3 = - \frac{a_0}{3 \cdot 2}$$

$k=1$

$$a_4 = - \frac{a_1}{4 \cdot 3}$$

$k=2$

$$a_5 = - \frac{a_2}{5 \cdot 4} = 0$$

$k=3$

$$a_6 = - \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$\vdots$$

$$a_{3n} = \left(- \frac{1}{3n(3n-1)}\right)\left(- \frac{1}{(3n-3)(3n-4)}\right) \cdots$$

$$\left(- \frac{1}{3 \cdot 2}\right) \cdot a_0$$

$$a_{3n+1} = \left(- \frac{1}{(3n+1)(3n)}\right)\left(- \frac{1}{(3n-2)(3n-3)}\right) \cdots$$

$$\left(- \frac{1}{4 \cdot 3}\right) \cdot a_1$$

$$a_{3n+2} = 0$$

So

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1}$$

$$+ \cancel{\sum_{n=0}^{\infty} \underbrace{a_{3n+2}}_{=0} x^{3n+2}} = 0$$

$$= a_0 \cdot \left(\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{3n}}{(3n(3n-1)) \cdot ((3n-3)(3n-4)) \cdots (3 \cdot 2)} \right)$$

$$+ a_1 \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{((3n+1)(3n)) \cdot ((3n-2)(3n-3)) \cdots (4 \cdot 3)} \right)$$

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