Calculus, week 1, Spring 2025

§ Sequence and limits with infinity

Recall

In Calculus, we consider 3 types of limits:

- function type A:

$$\lim_{x \to c} f(x) \quad (f: \text{real-valued functions})$$

- function type B:

We will also discuss this type in this section

$$\lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x)$$

(x: real numbers)

- sequence type:

$$\lim_{n \to \infty} a_n \quad (n: \text{integers})$$

main topic in this section

Def (Def 11.2.1) 數列，序列

A sequence of real numbers is a real-valued function defined on the set $\{1, 2, 3, \cdots\}$

That is, a sequence is of the form

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, a_4, \cdots$$

Example

① $a_n = n$, $n = 1, 2, 3, \cdots$ is a sequence:

$$1, 2, 3, 4, 5, \cdots\cdots$$

② Assume $a_1 = 1 = a_2$,

$$a_n = a_{n-1} + a_{n-2}, \quad n = 3, 4, \dots$$

This defines a sequence:

$a_4 = a_3 + a_2$

1, 1, 2, 3, 5, 8, 13, 21, ...

$a_3 = a_2 + a_1 = 1 + 1 = 2$

Operations for sequences

Let

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, \dots$$

$$(b_n)_{n=1}^{\infty} = b_1, b_2, b_3, \dots$$

be sequences (of real numbers)

and $r \in \mathbb{R}$ ← set of real numbers

in

One can form:

① scalar product sequence

① scalar product sequence

$$r \cdot (a_n)_{n=1}^{\infty} = (r \cdot a_n)_{n=1}^{\infty}$$

$$= ra_1,\ ra_2,\ ra_3,\ \cdots$$

② sum sequence:

$$(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty}$$

$$= (a_n + b_n)_{n=1}^{\infty}$$

$$= a_1+b_1,\ a_2+b_2,\ a_3+b_3,\ \cdots$$

③ difference sequence:

$$(a_n)_{n=1}^{\infty} - (b_n)_{n=1}^{\infty}$$

$$= (a_n)_{n=1}^{\infty} + (-1) \cdot (b_n)_{n=1}^{\infty}$$

$$= (a_n - b_n)_{n=1}^{\infty}$$

$$= (a_1-b_1),\ (a_2-b_2),\ \cdots\cdots$$

④ product sequence:

④ product sequence:

$$(a_n)_{n=1}^{\infty} \cdot (b_n)_{n=1}^{\infty}$$

$$= (a_n \cdot b_n)_{n=1}^{\infty}$$

$$= a_1 b_1,\ a_2 b_2,\ a_3 b_3,\ \ldots$$

Remark

$\leftarrow$ ①

$$r \cdot (a_n)_{n=1}^{\infty}$$

$$= (r, r, r, r, r, \ldots) \cdot (a_1, a_2, a_3, \ldots)$$ $\swarrow$ ④

for all

Now assume $b_n \neq 0 \quad \forall n$

⑤ reciprocal sequence:

$$\left(\frac{1}{b_n}\right)_{n=1}^{\infty}$$

$$= \frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}, \ldots$$

$b_1 \quad , \quad b_2 \quad , \quad b_3 \quad ,$

⑥ quotient sequence:

$$\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty} = (a_n)_{n=1}^{\infty} \cdot \left(\frac{1}{b_n}\right)_{n=1}^{\infty}$$

$$= \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$$

Some terminologies

We say that a sequence $(a_n)_{n=1}^{\infty}$ is

有上界

(i) bounded above if

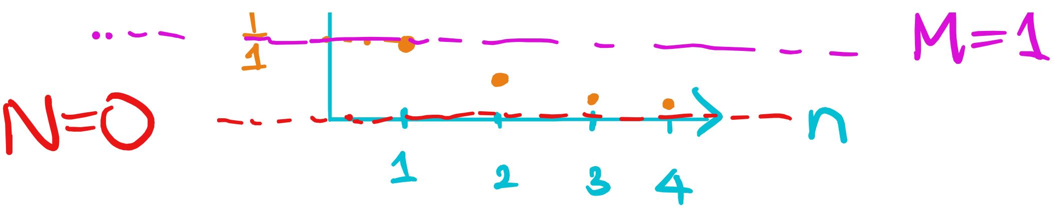
存在 such that

$\exists M$ s.t. $a_n \leq M \quad \forall n$

e.g. $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is bounded above

$(M = 1)$

$\frac{1}{n}$



(ii) 有下界 **bounded below** if

$\exists N$ s.t. $a_n \geq N \quad \forall n$

(iii) 有界 **bounded** if it is both bounded above and bounded below

(iv) 嚴格遞增 **strictly increasing** (= increasing in textbook)

if $a_n < a_{n+1} \quad \forall n$

(v) **increasing** = nondecreasing if

$a_n \leq a_{n+1} \quad \forall n$ in textbook

(vi) 嚴格遞減 **strictly decreasing** if

= decreasing in book

decreasing in book

$$a_n > a_{n+1} \quad \forall n$$

(vii) decreasing if = nonincreasing in book

$$a_n \geq a_{n+1} \quad \forall n$$

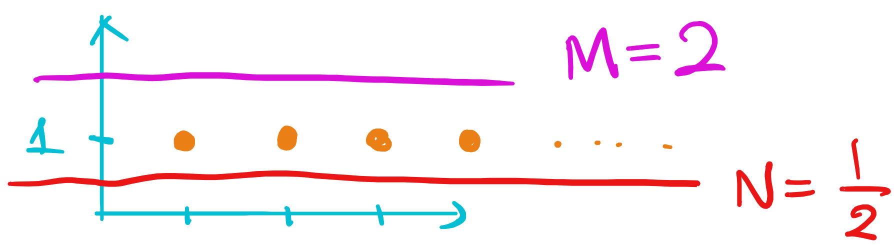
(viii) constant if

$$a_n = a_{n+1} \quad \forall n$$

Example

① $(1)_{n=1}^{\infty} = 1, 1, 1, 1, \ldots$

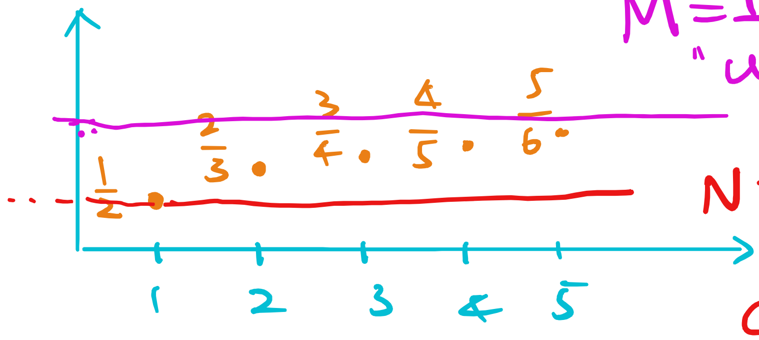
is a constant sequence and is bounded.



② $$\left(\frac{n}{n+1}\right)_{n=1}^{\infty} = \left(1 - \boxed{\frac{1}{n+1}}\right)_{n=1}^{\infty}$$

strictly decreasing

is strictly increasing
and bounded



$M=1$ is an "upper bound" 上界

$N=\frac{1}{2}$ is "a lower bound" 下界

③ $\left(\frac{n}{e^n}\right)_{n=1}^{\infty} =$ $(e=2.\cdots)$

an upper bound $\frac{1}{e}, \frac{2}{e^2}, \frac{3}{e^3}, \cdots$

is strictly decreasing
and bounded (0 is a lower bound)

Why strictly decreasing?

Consider

$$f(x) = \frac{x}{e^x}$$

$n = 1, 2, 3, \ldots$

Then

$$a_n = \frac{n}{e^n} = f(n)$$

$$\left(\frac{p(x)}{q(x)}\right)' = \frac{p' \cdot q - p \cdot q'}{q^2}$$

Since

$$f'(x) = \frac{(x)' \cdot e^x - x \cdot (e^x)'}{(e^x)^2}$$

$(e^x)' = e^x$

$(x^m)' = m x^{m-1}$

$$= \frac{1 \cdot \cancel{e^x} - x \cdot \cancel{e^x}}{(e^x)^{\cancel{2}}}$$

$$= \frac{1-x}{e^x} < 0$$

$e^x > 0$

$\forall \ x > 1$

we know

$$a_n = f(n), \ n = 1, 2, 3, \ldots$$

is strictly decreasing.

double check :

By the mean value thm

$$a_2 - a_1 = f(2) - f(1)$$

$$= (2-1) \cdot f'(c) \quad \text{for some } \underset{c>1}{c \in (1,2)}$$

$$< 0$$

Limits of sequences

<u>Def.</u>

A sequence $(a_n)_{n=1}^{\infty}$ is convergent (收斂的) if $\exists$ a number L with the property:

$\forall \varepsilon > 0$, $\exists$ K s.t.

$$|a_n - L| < \varepsilon \quad \text{when} \quad n \geq K$$

In this case, we say L is the limit of $(a_n)_{n=1}^{\infty}$, denoted

DY

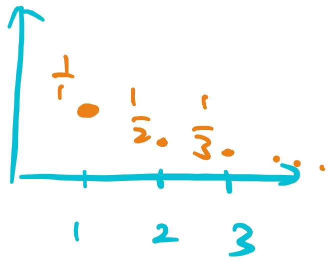
$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \quad (\text{as } n \to \infty)$$

發散

A sequence $(a_n)_{n=1}^{\infty}$ is divergent if it is not convergent.



<u>Example</u>

Prove $\lim_{n\to\infty} \frac{1}{n} = 0$.

<u>pf</u>

Given any $\varepsilon > 0$, we take

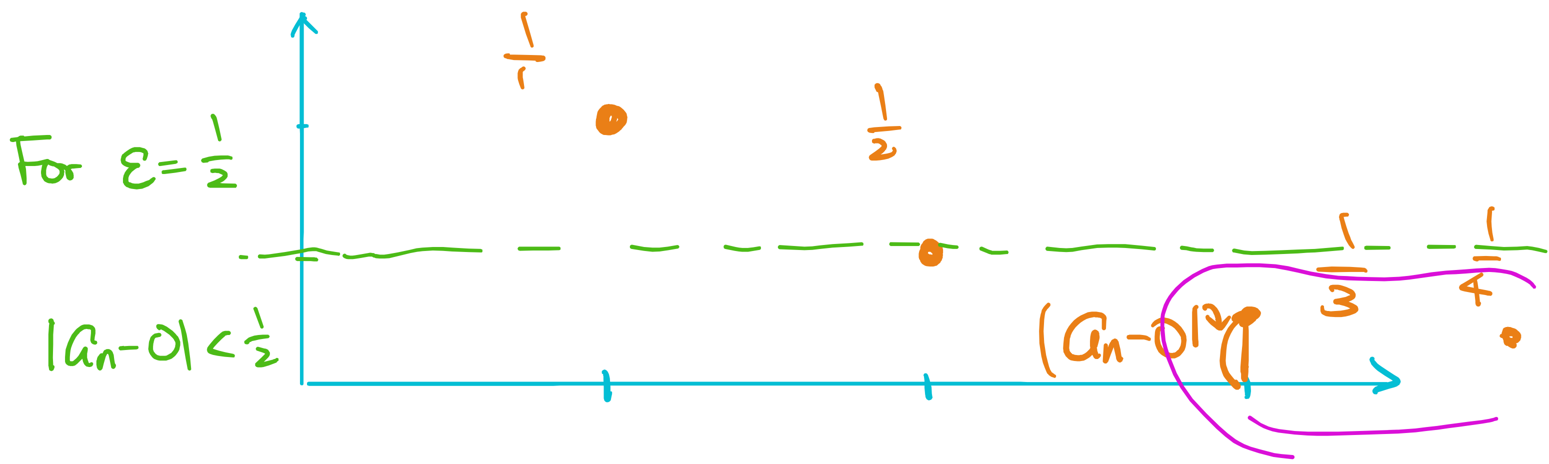
$$K = \frac{2}{\varepsilon}.$$

Then whenever $n \geq K$,

assume $K > 0$

$$\frac{1}{n} = \left|\frac{1}{n} - 0\right| \leq \frac{1}{K}$$

$$= \frac{1}{\cdots} = \frac{\varepsilon}{\cdots} < \varepsilon$$



<u>Thm</u> (§11.3)

(i) The limit is unique:

if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$

then $L = M$.

(ii) Every convergent sequence is bounded.

e.g. Since $(a_n = n)_{n=1}^{\infty} = 1, 2, 3, 4, \dots$

is unbounded, we know it is divergent.

(iii) Every unbounded sequence

is divergent

(iv) A bounded above increasing sequence is convergent.

(v) A bounded below decreasing sequence is convergent.

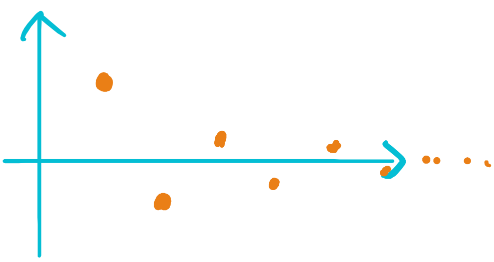
NOTE:

① $\left(a_n = (-1)^n\right)_{n=1}^{\infty} = -1, 1, -1, 1, -1, 1, \ldots$

is bounded and divergent

② $\left(a_n = \frac{(-1)^n}{n}\right)_{n=1}^{\infty} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

is convergent, not increasing
not decreasing



Pf of (ii)

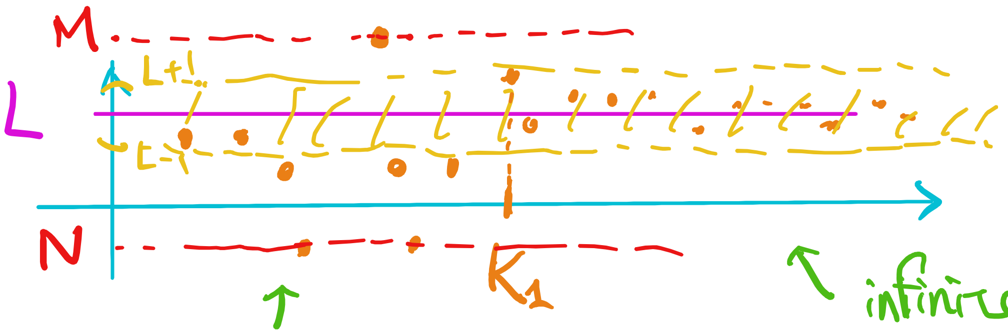
Assume $(a_n)_{n=1}^{\infty}$ converges to L.

That is, $\forall \varepsilon > 0 \quad \exists K = K_\varepsilon$ s.t.
whenever $n \geq K$,

$$|a_n - L| < \varepsilon$$

Consider $\varepsilon = 1$. $\exists K = K_1$ s.t.

$$|a_n - L| < 1 \qquad \text{when } n \geq K_1$$



only finite terms, so they are bounded

infinite terms but all of them are in $(L-1, L+1)$

Choose

$$M = \max\{a_1, a_2, \cdots, a_{K_1-1}, (L+1)\}$$

$$N = \min\{a_1, a_2, \cdots, a_{K_1-1}, (L-1)\}$$

Then

$$N \leq a_n \leq M \qquad \forall n = 1, 2, 3, \ldots$$

So $(a_n)_{n=1}^{\infty}$ is bounded. #

Example

Determine the sequence is convergent or divergent

① $(a_n = n)_{n=1}^{\infty}$ is unbounded $\Rightarrow$ divergent

② $(a_n = \frac{1}{n})_{n=1}^{\infty}$ is decreasing and bounded below by 0

$\Rightarrow$ it is convergent.

$1 - \frac{1}{n+1}$ "

③ $(a_n = \frac{n}{n+1})_{n=1}^{\infty}$ is increasing and bounded above by 1

$\Rightarrow$ it is convergent.

④ $(a_n = \frac{n}{e^n})_{n=1}^{\infty}$ is decreasing and bounded below by 0

$\Rightarrow$ it is convergent. #

Def

Def.

子數列

A subsequence of a sequence

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

is a sequence of the form

$$(a_{n_k})_{k=1}^{\infty} = a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

where $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers

e.g.

$$a_1, a_3, a_5, a_7, a_9, \ldots$$

is a subsequence of $(a_n)_{n=1}^{\infty}$

$$a_2, a_4, a_6, a_8, \ldots$$

and

$$a_1, a_2, a_3, a_5, a_8, a_{13}, a_{21}, \ldots$$

are subsequences of $(a_n)_{n=1}^{\infty}$

Thm

If a sequence $(a_n)_{n=1}^{\infty}$ converges to L

If a sequence converges to L, then all its subsequences converge to L.

pf

We know: $\forall \varepsilon > 0 \quad \exists \bar{K} = \bar{K}_\varepsilon$ s.t.

$$|a_n - L| < \varepsilon \quad \text{when} \quad n \geq \bar{K}_\varepsilon .$$

Let $(a_{n_k})_{k=1}^{\infty}$ be an arbitrary subseq. of $(a_n)_{n=1}^{\infty}$.

Given any $\varepsilon > 0$, choose the same $\bar{K} = \bar{K}_\varepsilon$.

Since $(n_k)_{k=1}^{\infty}$ is strictly increasing seq. of positive integers, we have

$$\begin{cases} \text{(i)} & n_1 \geq 1 \\ \text{(ii)} & n_{k+1} - n_k \geq 1 \end{cases}$$

$$\Rightarrow \quad n_2 = (n_2 - n_1) + n_1 \geq 1 + 1 = 2$$

$$n_3 = (n_3 - n_2) + n_2 \geq 1 + 2 = 3$$

$$\vdots$$

$n_k \geq k$

So if $k \geq \bar{K}_\varepsilon$, then $n_k \geq \bar{K}_\varepsilon$

$\Rightarrow |a_{n_k} - L| < \varepsilon$

That is,

$$\lim_{k\to\infty} a_{n_k} = L \qquad \#$$

Example

$(a_n = (-1)^n)_{n=1}^{\infty} = -1, 1, -1, 1, \dots$

The sequences a_{n_k}, $n_k = 2k-1$

$$\begin{array}{cccccc} k=1 & k=2 & k=3 & k=4 & k=5 & \\ -1, & -1, & -1, & -1, & -1, & \dots \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \\ a_1 & a_3 & a_5 & a_7 & a_9 & \end{array}$$

and a_{m_k}, $m_k = 2k$

$$\begin{array}{cccccc} k=1 & k=2 & k=3 & k=4 & k=5 & \\ 1, & 1, & 1, & 1, & 1, & \dots \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \\ a_2 & a_4 & a_6 & a_8 & a_{10} & \end{array}$$

are subsequences of $((-1)^n)_{n=1}^{\infty}$

Assume $(a_n = (-1)^n)_{n=1}^{\infty}$ converges to L.

Then

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1) = L$$

$$\lim_{k\to\infty} (-1) = -1$$

and

$$\lim_{k\to\infty} a_{m_k} = \lim_{k\to\infty} 1 = L$$

$$\lim_{k\to\infty} 1 = 1$$

$$-1 \neq 1$$

$(\rightarrow\times\leftarrow)$

So $(a_n = (-1)^n)_{n=1}^{\infty}$ is divergent.

<u>Remark</u>

If all the subsequences of $(a_n)_{n=1}^{\infty}$ converges to L, then

$$\lim_{n\to\infty} a_n = L$$

Take $n_k = k$: $(a_n)_{n=1}^{\infty}$ is also a subseq of $(a_n)_{n=1}^{\infty}$

Computation of $\lim_{n\to\infty} a_n$

Thm (Thm 11.3.7)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences, and $r \in \mathbb{R}$. Then

(i) $\lim\limits_{n\to\infty} (r \cdot a_n) = r \cdot \left(\lim\limits_{n\to\infty} a_n\right)$

(ii) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

(iii) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$

(iv) If $b_n \neq 0$ $\forall n$ and $\lim\limits_{n\to\infty} b_n$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

This is to make sense $\dfrac{a_n}{b_n}$

If $b_n = 0$ for finite n, then we can consider $b_M, b_{M+1}, b_{M+2}, \cdots$ s.t. $b_n \neq 0$ $\forall n \geq M$

Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ still holds.

Example

Recall $\lim_{n\to\infty} \frac{1}{n} = 0$

Compute the limits.

① $\lim_{n\to\infty} \frac{(3n^4-2n^2+1)\times\frac{1}{n^5}}{(n^5-3n^3)\times\frac{1}{n^5}}$

$$= \lim_{n\to\infty} \frac{3\cdot\frac{1}{n} - 2\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^5}{1 - 3\cdot\left(\frac{1}{n}\right)^2}$$

$$= \frac{3\cdot\big(\underbrace{\cancel{\lim_{n\to\infty}\tfrac{1}{n}}}_{=0}\big) - 2\cdot\big(\underbrace{\cancel{\lim_{n\to\infty}\tfrac{1}{n}}}_{=0}\big)^3 + \big(\underbrace{\cancel{\lim_{n\to\infty}\tfrac{1}{n}}}_{=0}\big)^5}{1 - 3\big(\underbrace{\cancel{\lim_{n\to\infty}\tfrac{1}{n}}}_{=0}\big)^2}$$

$$= \frac{0}{1} = 0 \quad \#$$

② $\lim_{n\to\infty} \frac{(1-4n^7)\times\frac{1}{n^7}}{(n^7+12n)\times\frac{1}{n^7}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^7} - 4}{1 + 12\cdot\frac{1}{n^6}}$$

$0^7 = 0$

$0^6 = 0$

$$= \frac{-4}{1} = -4 \quad \#$$

③ $\lim$ $n^4 - 3n^2 + n + 2$

0

$-3\cdot\frac{1}{n} + 4\frac{1}{n^2} + \frac{2}{n^3}$

③ $\lim_{n\to\infty} \frac{[illegible]}{n^3 - 7n}$

$1 - 7\frac{1}{n^2}$

$\parallel$

$$= \lim_{n\to\infty}\left(\frac{n^4 \times \frac{1}{n^3}}{(n^3-7n)\times\frac{1}{n^3}} + \frac{\frac{1}{n^3}\cdot(-3n^2+n+2)}{\frac{1}{n^3}\cdot(n^3-7n)}\right)$$

bounded

$\parallel$

$$\frac{n}{1-7\frac{1}{n^2}} > \frac{n}{1} \quad \text{for } n \geq 3$$

no upper bound

For $n \geq 3$, $0 < 1 - 7\frac{1}{n^2} < 1$

Since $(n)_{n=1}^{\infty}$ has no upper bound,

$\frac{n}{1-7\frac{1}{n^2}}$ has no upper bound either

So $\frac{n^4}{n^3-7n} + \frac{-3n^2+n+2}{n^3-7n}$

has no upper bound

$\Rightarrow$ it is divergent. #

夾擠

Thm (Pinching thm ... Thm 1.39)

Thm (Pinching thm for seq., [illegible])

Suppose that for n sufficiently large, we have

↖ That is, $\exists M$ s.t.

$$a_n \le b_n \le c_n \quad \forall n \ge M.$$

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then

$$\lim_{n\to\infty} b_n = L.$$

Example

① $\displaystyle\lim_{n\to\infty} \frac{\cos n}{n} = ?$

Since

$$-1 \le \cos n \le 1$$

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

and

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0 = \lim_{n\to\infty} \frac{1}{n}$$

by Pinching Thm,

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0 \quad \#$$