Calculus (I) — Homework 9 (Fall 2024)

1. Calculate.

   (a) $\int_0^1 x(x^2+1)^3\,dx$.

   (b) $\int_{-1}^0 3x^2(4+2x^3)^2\,dx$.

   (c) $\int_0^1 x\sqrt{x+1}\,dx$.

   (d) $\int_0^1 \frac{x+3}{\sqrt{x+1}}\,dx$.

   (e) $\int_{-\pi}^{\pi} \sin^4 x\cos x\,dx$.

   (f) $\int_0^1 \cos^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}x)\,dx$.

   (g) $\int_0^{\pi} x\cos x^2\,dx$.

   (h) $\int_0^{\pi} x^2\cos x\,dx$.

   (i) $\int_0^{\pi/2} \cos^2 2x\,dx$.

   (j) $\int_0^{2\pi} \sin^2 x\,dx$.

   (k) $\int_0^1 x(x+5)^{14}\,dx$.

   (l) $\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx$.

   (m) $\int_0^{\pi/2} \cos(\sqrt{x})\,dx$.

2. Calculate.

   (a) $\frac{d}{dx}\left(\int_0^{1+x^2} \frac{dt}{\sqrt{2t+5}}\right)$.

   (b) $\frac{d}{dx}\left(\int_{3x}^{1/x} \cos 2t\,dt\right)$.

   (c) $\int_{-3}^{3} \frac{t^3}{1+t^2}\,dt$.

   (d) $\int_{-\pi/4}^{\pi/4} (x^2-2x+\sin x+\cos 2x)\,dx$.

3. Let f be a continuous function, and a, b, c be real numbers.

   (a) Show that
   $$\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx.$$

   (b) Show that, if $c \neq 0$,
   $$\frac{1}{c}\int_{ac}^{bc} f(x/c)\,dx = \int_a^b f(x)\,dx.$$

4. True or false. Explain your answers. Assume f and g are continuous on $[a,b]$, $a < b$.

   (a) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $\int_a^b [f(x)-g(x)]\,dx > 0$.

   (b) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for all $x \in [a,b]$.

   (c) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for some $x \in [a,b]$.

   (d) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $\int_a^b |f(x)|\,dx > \int_a^b |g(x)|\,dx$.

   (e) If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

   (f) If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for some $x \in [a,b]$.

   (g) If $\int_a^b |f(x)|\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

   (h) If $\int_a^b f(x)\,dx = 0$, then $\int_a^b |f(x)|\,dx = 0$.

Remark. *In the following problems, we assume that the domain is the maximal possible domain in the set of real numbers unless otherwise stated, and assume that the codomain is the range of the function.*

5. Determine whether or not the function is one-to-one. If the function has an inverse, find it and sketch the graphs of the function and its inverse.

   (a) $f(x) = 5x + 3, \quad x \in (-\infty, \infty)$.
   (b) $f(x) = 1 - x^2, \quad x \in (-\infty, \infty)$.
   (c) $f(x) = \sin x, \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
   (d) $f(x) = \cos x, \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

6. Verify that f has an inverse and find $(f^{-1})'(c)$.

   (a) $f(x) = x^3 + 1, x \in (-\infty, \infty);\ c = 9$.
   (b) $f(x) = \sin x, -\frac{1}{2}\pi < x < \frac{1}{2}\pi;\ c = -\frac{1}{2}$.
   (c) $f(x) = \dfrac{x+3}{x-1}, x > 1;\ c = 3$.
   (d) $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\, dt, x \in (-\infty, \infty);\ c = 0$.

7. Set
$$f(x) = \int_2^x \sqrt{1+t^2}\, dt.$$
   (a) Show that f has an inverse.
   (b) Find $(f^{-1})'(0)$.

8. Estimate the values.

   (a) $\ln 20$.
   (b) $\ln 16$.
   (c) $\ln 3^4$.

   ($\ln 2 \approx 0.7, \ln 3 \approx 1.1, \ln 5 \approx 1.6$.)

9. Show that
$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

10. Determine the domain and find the derivative.

   (a) $f(x) = (\ln x)^3$.
   (b) $f(x) = \ln(\ln x)$.
   (c) $f(x) = \dfrac{1}{\ln x}$.
   (d) $f(x) = \ln\left|\dfrac{x+2}{x^3-1}\right|$.
   (e) $f(x) = \sin(\ln x)$.

11. Show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. (Hint: First show the inequality $\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$ by noting $1 < \frac{1}{t} < \frac{x}{1+x}$ for $t \in (1, 1+\frac{1}{x})$.)

12. Evaluate.

   (a) $\displaystyle\int_1^e \frac{dx}{x}$.
   (b) $\displaystyle\int_e^{e^2} \frac{dx}{x}$.
   (c) $\displaystyle\int_4^5 \frac{x}{x^2-1}\, dx$.
   (d) $\displaystyle\int_{1/4}^{1/3} \tan(\pi x)\, dx$.
   (e) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin x}\, dx$.
   (f) $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$.
   (g) $\displaystyle\int_1^2 \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx$.
   (h) $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$.