# Calculus (I) — Homework 6 (Fall 2024)

1. Prove that if $|f(x) - (3x + 2)| \leq |x|^{\frac{3}{2}}$ for any real numbers $x$, then $f$ is differentiable at 0, and $y = 3x + 2$ is the tangent line of the graph of $f$ at $(0, f(0))$. (Approximation of $f$ by a line around $x = 0$.)

2. (i) Determine whether or not $f$ satisfies the conditions of the mean value theorem on the indicated interval $[a, b]$. (ii) Find all the numbers $c$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

   (a) $f(x) = x^3 - x$; $[0, 1]$.
   (b) $f(x) = x^2$; $[1, 2]$.
   (c) $f(x) = 3\sqrt{x} - 4x$; $[1, 4]$.
   (d) $f(x) = \sin x$; $[0, \pi]$.
   (e) $f(x) = \sqrt{1 - x^2}$; $[0, 1]$.
   (f) $f(x) = x^{2/3} - 1$; $[-1, 1]$.

3. Suppose that $f$ is differentiable on $(2, 6)$ and continuous on $[2, 6]$. Given that $1 \leq f'(x) \leq 3$ for all $x$ in $(2, 6)$, show that
$$4 \leq f(6) - f(2) \leq 12.$$

4. Prove that for all real numbers $x$ and $y$
   (a) $|\cos x - \cos y| \leq |x - y|$.
   (b) $|\sin x - \sin y| \leq |x - y|$.

5. Suppose that $f''$ is continuous on $[a, b]$ and that $f$ has 3 distinct zeros in $[a, b]$. Prove that $f''$ has at least one zero in $(a, b)$.

6. Find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing.

   (a) $f(x) = x^3 - 3x + 2$.
   (b) $f(x) = x + \frac{1}{x}$.
   (c) $f(x) = |x^2 - 5|$.
   (d) $f(x) = x - \cos x$.

7. Show that
$$\tan x > x,$$
   for all $x$ in $(0, \frac{\pi}{2})$.

8. True or false? Explain your answers.
   (a) The function $f(x) = x^2$ is an increasing function on $(-\infty, \infty)$.
   (b) The function $f(x) = x^2$ is a decreasing function on $(-\infty, \infty)$.
   (c) The function $f(x) = x^2$ is an increasing function on $(0, \infty)$.
   (d) The function $f(x) = x^3$ is an increasing function on $(-\infty, \infty)$.

9. Suppose a function $f$ has derivative
$$f'(x) = x^3(x - 1)^2(x + 1)(x - 2).$$
   At what number(s) $x$, if any, does $f$ have a critical point? A local maximum? A local minimum?