Calculus (I) — Homework 3 (Fall 2024)

1. Determine whether or not the function is continuous at the indicated point. Explain your answers.

   (a) $f(x) = x^3 - 5x + 1\,, \quad x = 2.$

   (b) $f(x) = x\sin x + \cos^2 x\,, \quad x = 1.$

   (c) $f(x) = \tan x\,, \quad x = \pi/2.$

   (d) $f(x) = \sqrt{(x-1)^2 + 5}\,, \quad x = 1.$

   (e) $f(x) = |4 - x^2|\,, \quad x = 2.$

   (f) $f(x) = \begin{cases} x^2 + 4, & x < 2, \\ x^3, & x \geq 2, \end{cases} \quad x = 2.$

   (g) $f(x) = \begin{cases} x^2 + 5, & x < 2, \\ x^3, & x \geq 2, \end{cases} \quad x = 2.$

   (h) $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1, \\ 0, & x = 1, \end{cases} \quad x = 1.$

   (i) $f(x) = \begin{cases} -x^2, & x < 0, \\ 0, & x = 0, \\ 1/x^2, & x > 0, \end{cases} \quad x = 0.$

2. Sketch the graph and classify the discontinuities (if any) as being removable or essential.

   (a) $f(x) = |x^2 - 1|\,.$

   (b) $f(x) = \tan(x+1)\,.$

   (c) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2, \\ 0, & x = 2. \end{cases}$

   (d) $f(x) = \begin{cases} \dfrac{x + 2}{x^2 - x - 6}, & x \neq -2, 3, \\ -\frac{1}{5}, & x = -2, 3. \end{cases}$

   (e) $f(x) = \begin{cases} \sin x \cos x, & x < 0, \\ 0, & x = 0, \\ 1/x^2 & x > 0. \end{cases}$

3. Evaluate the limits.

   (a) $\lim\limits_{x\to\pi} \sin(x - \sin x).$

   (b) $\lim\limits_{x\to 0} \sin\Big(\dfrac{\pi}{2}\cos(\tan x)\Big).$

   (c) $\lim\limits_{x\to 0} \cos\Big(\dfrac{\pi}{\sqrt{19 - 3\sec(2x)}}\Big).$

   (d) $\lim\limits_{x\to 0^+} \sin\Big(\dfrac{\pi}{2}\cos(\sqrt{x})\Big).$

4. Use the intermediate value theorem to show that there is a solution of the given equation in the indicated interval.

   (a) $2x^3 - 4x^2 + 5x - 4 = 0, \quad [1, 2].$

   (b) $\sin x + 2\cos x - x^2 = 0, \quad [0, \pi/2].$

5. (Brouwer fixed-point theorem.) Show that if f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$ for all x in $[0, 1]$, then there exists at least one point c in $[0, 1]$ at which $f(c) = c$. (HINT: Apply the intermediate value theorem to the function $g(x) = x - f(x)$.)

6. True or false? Explain how your answers are consistent with the extreme value theorem

   (a) The function $f(x) = x^2$ attains a maximum value on $[-1, 1]$.

   (b) The function $f(x) = x^2$ attains a minimum value on $[-1, 1]$.

   (c) The function $f(x) = x^2$ attains a maximum value on $(-1, 1)$.

   (d) The function $f(x) = x^2$ attains a minimum value on $(-1, 1)$.

   (e) The function $f(x) = x^2$ is bounded on $(-1, 1)$.