Calculus, week 5, Fall 24

§ Derivatives of trigonometric functions

Thm (Eq. (3.6.1), (3.6.1))

The functions $\sin x$ and $\cos x$ are differentiable at any x.

Furthermore,

① $$(\sin x)' = \cos x$$

② $$(\cos x)' = -\sin x$$

pf

Recall that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Recall

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x}{h}$$

By Definition,

和角公式

$$① \ (\sin x)' = \lim_{h\to 0}\frac{\overset{\sin x\cos h + \cos x\sin h}{\overset{\parallel}{\boxed{\sin(x+h)}}} - \boxed{\sin x}}{h}$$

$$= \lim_{h\to 0}\left(\sin x\cdot \overset{0}{\overset{\nearrow}{\boxed{\frac{\cos h - 1}{h}}}} + \cos x\cdot \overset{1}{\overset{\nearrow}{\boxed{\frac{\sin h}{h}}}}\right)$$

$$= (\sin x)\cdot 0 + (\cos x)\cdot 1$$

$$= \cos x$$

和角公式

$$② \ (\cos x)' = \lim_{h\to 0}\frac{\overset{\boxed{\cos x\cos h} - \sin x\sin h}{\overset{\parallel}{\boxed{\cos(x+h)}}} - \boxed{\cos x}}{h}$$

$$= \lim_{h\to 0}\left(\cos x\cdot \overset{0}{\overset{\uparrow}{\boxed{\frac{\cos h - 1}{h}}}} - \sin x\cdot \overset{1}{\overset{\uparrow}{\boxed{\frac{\sin h}{h}}}}\right)$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x \qquad \#$$

Recall

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

✓ $f = \sin x$

$g = \cos x$

Remark

③ $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{\overset{\cos x =}{\boxed{(\sin x)'}} \cdot \cos x - (\sin x) \cdot \underset{= -\sin x}{\boxed{(\cos x)'}}}{(\cos x)^2}$$

$$= \frac{\overset{= 1}{\boxed{(\cos x)^2 + (\sin x)^2}}}{(\cos x)^2}$$

$$= \frac{1}{\cos^2 x} = \boxed{\begin{matrix} \sec^2 x \\ \parallel \\ (\tan x)' \end{matrix}}$$

④ $(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \cdots$

$$= -\csc^2 x$$

Recall

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

⑤ $(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-(\cos x)'}{(\cos x)^2}$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x$$

⑥ $(\csc x)' = \left(\frac{1}{\sin x}\right)' = \cdots$

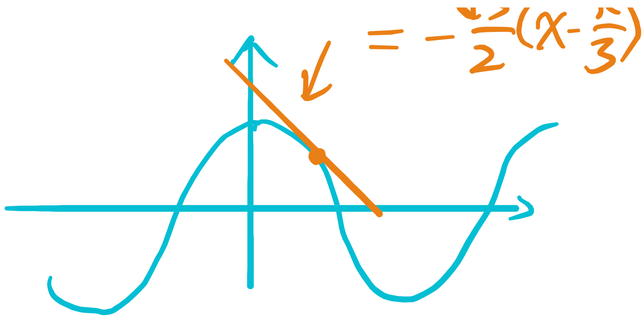
$$= -\csc x \cdot \cot x.$$

Example

Find the tangent line of $y - \frac{1}{2}$

$$y = \cos x$$

at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.



Sol

$$(\cos x)'\Big|_{x=\frac{\pi}{3}} = -\sin x\Big|_{x=\frac{\pi}{3}}$$

$$= -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

So the tangent line is

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$ #

Example

Find the derivatives of

① $\cos(2x)$

② $\sec(x^2+1)$

Recall (Chain rule)

$Z = Z(y)$

③ $\sin(x^{\circ})$

$y = y(x)$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

<u>Sol</u>

① Let $y = 2x$, $z = \cos y$

$$\underset{\| \ dy}{d \cos y} = -\sin y$$

$$\frac{d\cos(2x)}{dx} = \frac{dz}{dx} = \boxed{\frac{dz}{dy}} \cdot \boxed{\frac{dy}{dx}}$$

$$\| \quad \frac{d(2x)}{dx} = 2$$

$$= (-\sin y) \cdot 2$$

$$= -2\sin(2x) \ \#$$

② $$\left(\sec(x^2+1)\right)' = \left(\frac{1}{\cos(x^2+1)}\right)'$$

$$= -\frac{\left(\cos(x^2+1)\right)'}{\cos^2(x^2+1)}$$

$$= -\frac{-\sin(x^2+1) \cdot 2x}{\cos^2(x^2+1)}$$

$$\sin(x^2+1)$$

$$= \frac{2x}{\cos^2(x^2+1)}$$

$$= 2x \cdot \sec(x^2+1)\cdot \tan(x^2+1) \quad \#$$

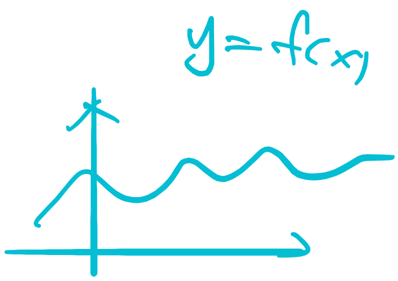
③ $$(\sin x^\circ)' = \left(\sin\left(\frac{x}{360}\cdot 2\pi\right)\right)'$$

$$= \left(\overbrace{\sin\left(\overset{\overset{y}{\shortparallel}}{\underline{\frac{2\pi}{360}\cdot x}}\right)}^{z}\right)'$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \underset{\cos x^\circ}{\underline{\overset{\frac{dz}{dy}}{\cos}\left(\frac{2\pi}{360}\cdot x\right)}} \cdot \overset{\frac{dy}{dx}}{\overset{\overset{\frac{2\pi}{360}}{\shortparallel}}{\boxed{\left(\frac{2\pi}{360}x\right)'}}}$$

$$= \frac{\pi}{180}\cdot \cos x^\circ \quad \#$$

§ Implicit differentiation

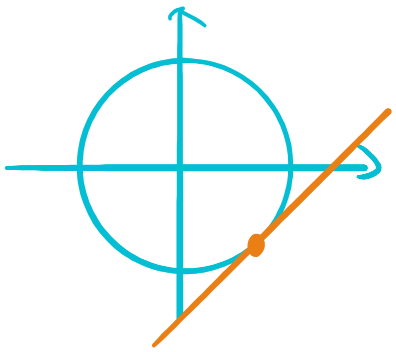


Example

Find the tangent line of

$$x^2+y^2=1$$

at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$



<u>sol</u>

Differentiate the both sides of the equation:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(1) = 0$$

$$= \overset{\overset{2X}{\shortparallel}}{\underline{\frac{d}{dx}(x^2)}} + \underline{\frac{d}{dx}(y^2)}$$

$z = y^2$

$y = y(x)$

$$= \frac{dz}{dy}\cdot\frac{dy}{dx}$$

$$= 2y\cdot\frac{dy}{dx}$$

So

$$2X + 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$$\Rightarrow \quad \frac{dy}{dx}\bigg|_{(x,y)=\left(\frac{1}{\sqrt{2}},\,-\frac{1}{\sqrt{2}}\right)}$$

$$= \left(-\frac{x}{y}\right)\bigg|_{\left(\frac{1}{\sqrt{2}},\,-\frac{1}{\sqrt{2}}\right)}$$

$$= -\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$$

$\Rightarrow$ The tangent line is

$$\left(y - \left(-\frac{1}{\sqrt{2}}\right)\right) = 1\cdot\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\shortparallel$$

$y + \frac{1}{\sqrt{2}}$ #

Example

Find tangent line of

$$2x^3 + 2y^3 = 9xy$$

at (1, 2).

sol

$$\frac{d}{dx}(2x^3 + 2y^3) = \frac{d}{dx}(9xy)$$

By Chain Rule,

$$\frac{d}{dx}(2x^3 + 2y^3) = 6x^2 + 6y^2 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(9xy) = 9y + 9x\frac{dy}{dx}$$))

$$\Rightarrow \quad 6y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} = 9y - 6x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{9y-6x^2}{6y^2-9x}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{9\times 2 - 6\times 1^2}{6\times 2^2 - 9\times 1} = \frac{4}{5}$$

⇒ The tangent line is

$$(y-2) = \frac{4}{5}(x-1) \qquad \#$$

Example

Let

$$\cos(x-y) = xy$$

Find $\frac{dy}{dx}$.

Sol

$$\frac{d}{dx}\left(\cos(x-y)\right) = \frac{d}{dx}(xy)$$

$$\frac{d}{dx}\left(\cos(x-y)\right) \underset{\text{rule}}{\overset{\text{chain}}{=}} -\sin(x-y)\cdot\frac{d}{dx}(x-y)$$

$$= \underline{-\sin(x-y)\cdot\left(1-\frac{dy}{dx}\right.}\left.\vphantom{\frac{dy}{dx}}\right)$$

$$/\!/$$

$$\frac{d}{dx}(xy) = \frac{dx}{dx}\cdot y + x\cdot\frac{dy}{dx} = \underline{y + x\frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx}\cdot\left(\sin(x-y) - x\right)$$

$$= y + \sin(x-y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+\sin(x-y)}{\sin(x-y) - x} \quad \#$$

Example

Differentiate (for $x>0$)

① $x^{\frac{1}{n}}$

② $x^{\frac{m}{n}}$

③ $\sqrt{\frac{x}{1+x^2}}$

<u>Recall</u>

$(x^n)' = n.x^{n-1}$

<u>Sol</u>

① Let $y = x^{\frac{1}{n}} \Rightarrow y^n = x$

$$\Rightarrow \frac{d}{dx}(y^n) = \frac{d}{dx}(x) = \underline{1}$$

|| chain rule

$$\underline{n.y^{n-1} . \frac{dy}{dx}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{n.y^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}}$$

|| $x^{\frac{n-1}{n}}$

$$= 1 . x^{-\frac{n-1}{n}}$$

|| $1-\frac{1}{n}$

n

$$= \frac{1}{n} \cdot x^{\frac{1}{n}-1} = \left(x^{\frac{1}{n}}\right)'$$ #

② $$\left(x^{\frac{m}{n}}\right)' = \left(\left(x^{\frac{1}{n}}\right)^m\right)'$$ Let $y = x^{\frac{1}{n}}$

$$= \frac{d\,y^m}{dx} \overset{\text{Chain}}{\underset{\text{rule}}{=}} m \cdot y^{m-1} \cdot \frac{dy}{dx}$$

$$= m \cdot \left(x^{\frac{1}{n}}\right)^{m-1} \cdot \boxed{\left(x^{\frac{1}{n}}\right)'} = \frac{1}{n} \cdot x^{\frac{1}{n}-1}$$

$$= \frac{m}{n} \cdot x^{\frac{m-\cancel{1}}{n} + \frac{\cancel{1}}{n} - 1}$$

$$= \frac{m}{n} \cdot x^{\frac{m}{n}-1} = \left(x^{\frac{m}{n}}\right)'$$ #

③ $\left(\sqrt{}\right)' \quad \left(\left(\ x\ \right)^{\frac{1}{2}}\right)'$

$$\left(\sqrt{\frac{x}{1+x^2}}\right) = \left(\left(\frac{\ }{1+x^2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{x}{1+x^2}\right)^{\frac{1}{2}-1} \cdot \left(\frac{x}{1+x^2}\right)'$$

$$= \frac{1}{2}\sqrt{\frac{1+x^2}{x}} \cdot \frac{\overset{1}{\overset{\|}{\boxed{(x)'}}}\cdot(1+x^2) - x\cdot\overset{2x}{\boxed{(1+x^2)'}}}{(1+x^2)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1+x^2}{x}} \cdot \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{1}{2}\frac{1-x^2}{\sqrt{x}\ (1+x^2)^{\frac{3}{2}}}$$

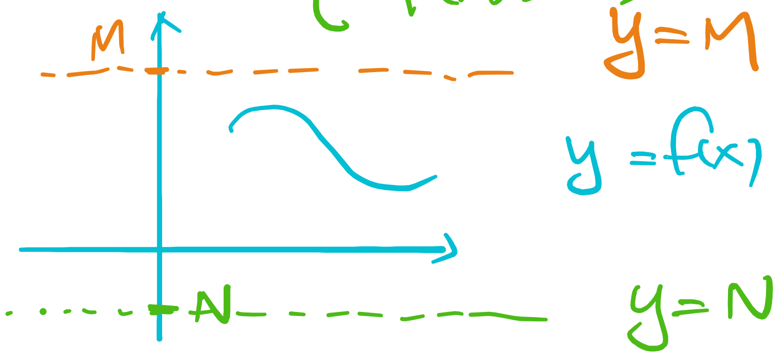
#

<u>Recall</u> (Boundedness, maximum, minimum)

A function f is <u>bounded above</u> 有上界

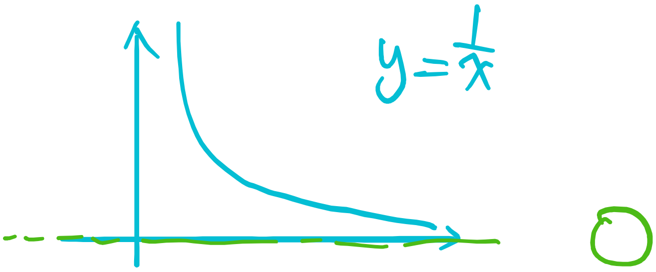
on $[a,b]$ if $\exists\ M$ ($\exists\ N$) s.t. (respectively below) 有下界

$$f(x) \le M \quad (f(x) \ge N) \quad \forall\ x \in [a,b]$$



Example

① $y=\frac{1}{x}$, $x\in(0,+\infty)$, is bounded below but NOT bounded above

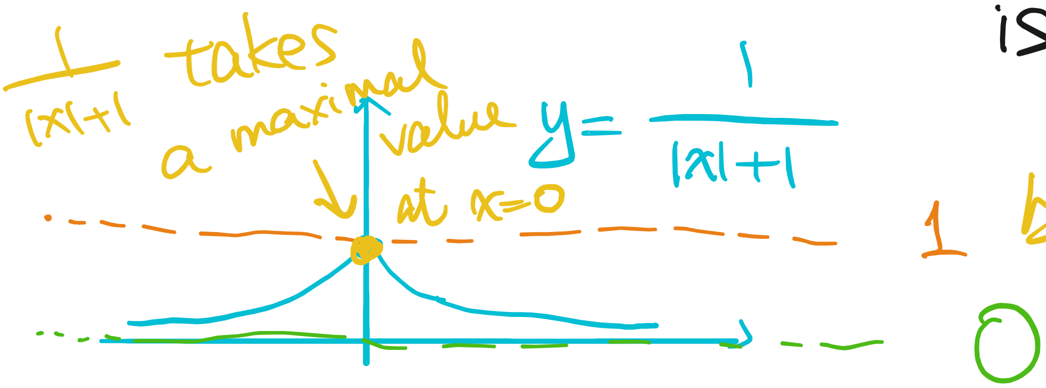


② $y=\frac{1}{|x|+1}$, $x\in(-\infty,\infty)$ 有界

is bounded

||

bounded above and



But it does NOT take
a minimal value
at any points

and
bounded
below