Calculus, week 3, Fall 24

§ Continuity 連續

Def (Def 2.4.1, Def 2.4.5) in

Let f be a function, $c \in \mathbb{R}$. {real numbers}

The function f is continuous at c if f is defined in $(c-p, c+p)$ for some $p>0$, and

$$\lim_{x \to c} f(x) = f(c).$$

We say f is discontinuous at c if f is not continuous at c.

The function f is right-continuous (left-continuous) at c (or continuous from the right (left)) if f is defined in $[c, c+p)$ (in $(c-p, c]$)

for some $p>0$, and

$$\lim_{x \to c^+} f(x) = f(c).$$

(in $(c-p, c]$)

$$\left(\lim_{x \to c^-} f(x) = f(c) \right)$$

The function f is continuous on the open interval (a,b) if f is continuous at each point $c \in (a,b)$.

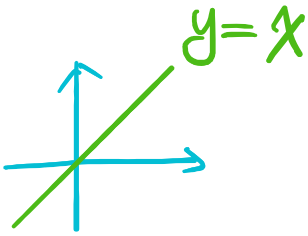
The function f is continuous on the closed interval $[a,b]$ if

(i) f is continuous on (a,b),
(ii) f is right-continuous at a,
(iii) f is left-continuous at b.

The concepts that f is continuous on $(a,b]$, $[a,b)$, $(-\infty,b]$, $[a,\infty)$, or $(-\infty,\infty)$ can be defined or $(-\infty, \infty)$ can be defined.

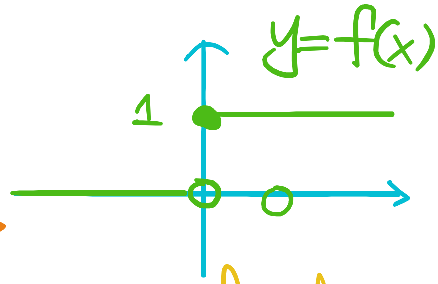
Example

① $f(x) = x$



is continuous on $(-\infty, \infty)$

② $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$



essential discontinuity

NOTE: This type of discontinuity is also called a jump discontinuity

$$\lim_{x \to 0^-} f(x) = 0 \neq \lim_{x \to 0^+} f(x) = 1 = f(0)$$

$\lim_{x \to 0} f(x)$ does NOT exist.

We conclude that

(i) f is discontinuous at 0

(ii) f is right-continuous at 0

Remark

A function f is continuous at c

iff $\leftarrow$ if and only if

(i) $f(c)$ exists.

(ii) $\lim_{x \to c} f(x)$ exists,

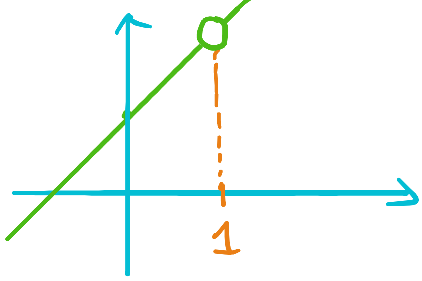
(iii) $f(c) = \lim_{x \to c} f(x)$

Example

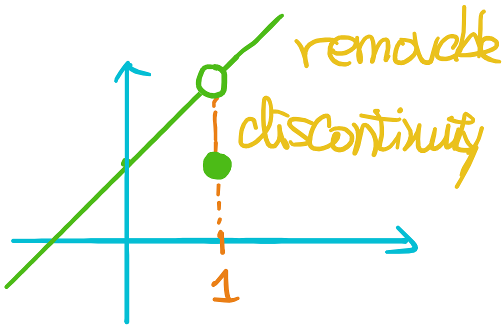
③ $f(x) = \dfrac{x^2 - 1}{x - 1}$ is discontinuous at 1 $\qquad x^2-1 = (x-1)(x+1)$

because $f(1)$ is NOT defined

$$= \begin{cases} \text{NOT defined}, & x = 1 \\ x+1 & x \neq 1 \end{cases}$$



④ $f(x) = \begin{cases} x+1, & x \neq 1 \\ 1, & x = 1 \end{cases}$



is discontinuous at 1 because

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x+1) = 2 \neq 1 = f(1)$$

Remark

- Any polynomials, $|x|$, $\sin x$, $\cos x$ are continuous on $(-\infty, \infty)$.
- The function $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$.

Def

If $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) \neq f(c)$, then we say that f has a removable discontinuity at c.

If $\lim_{x\to c} f(x)$ does NOT exist, then we say that f has an essential discontinuity at c.

If $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist,

$x \to c^+$ $\qquad\qquad$ $x \to c$

and
$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x),$$
then we say f has a jump discontinuity at c

Thm (Thm 2.4.2)

If f, g are continuous at c, and k is a constant, then
$$f+g, \quad f-g, \quad k \cdot f, \quad f \cdot g$$
are continuous at c.

Furthermore, if $g(c) \neq 0$, then $\frac{f}{g}$ is also continuous at c.

Example

Determine the continuity of

$$f(x)=\begin{cases} 3|x|+\dfrac{x^3-x}{x^2-5x+6} & \text{if } x\neq 2,3,\\ 0 & \text{if } x=2,3\end{cases}$$

<u>Sol</u>

<u>NOTE</u>: $x^2-5x+\overset{(-2)\times(-3)}{\underline{6}} = (x-2)(x-3)$

Since

$|x|,\quad x^3-x,\quad x^2-5x+6$

are continuous, by Thm, the function

f is continuous at any $c\neq 2,3$

Since

$2^3-2=6\neq 0$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\underbrace{3|x|}_{\to 6}+\frac{x^3-x}{\underbrace{x^2-5x+6}_{\to 0}}\right) \quad \text{does NOT exist}$$

$3^3-3=24\neq 0$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\left(\underbrace{3|x|}_{\to 9}+\frac{x^3-x}{\underbrace{x^2-5x+6}_{\to 0}}\right) \quad \text{does NOT exist}$$

we conclude that f has essential discontinuities at $c=2,3$. #

Thm (Thm 2.4.4)

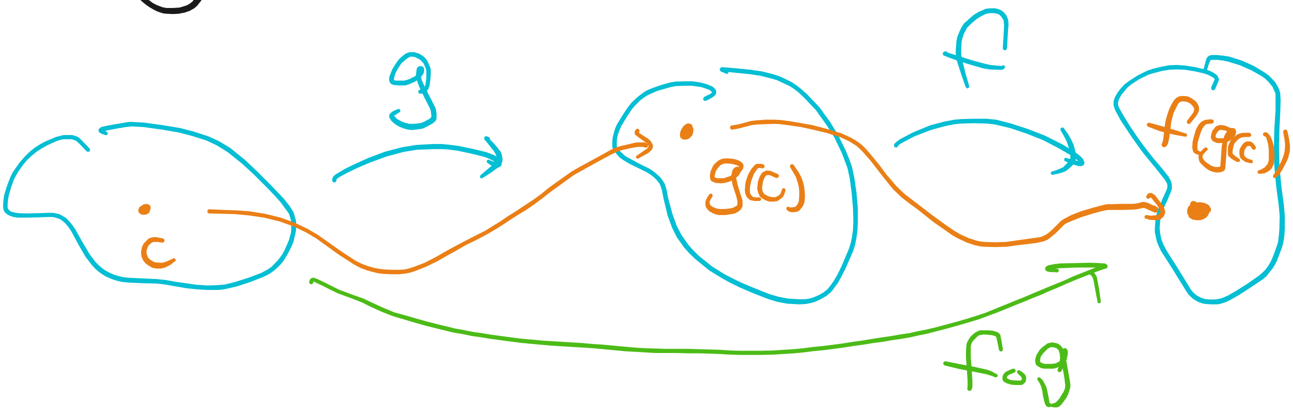
If

g is continuous at c, and

f is continuous at $g(c)$

then

$f \circ g$ is continuous at c.



Example

① $f(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous on $(3, \infty)$

because $f = f_1 \circ f_2$, where

$f_1(x) = \sqrt{x}$, $\qquad f_2(x) = \dfrac{x^2+1}{x-3}$

continuous on $[0, \infty)$ $\qquad$ continuous at any $c \neq 3$

For $x \in (3, \infty)$,

$x^2+1 > 0$

$f(x) \qquad > 0$

$f_2(x) = \sqrt{x-3}$ >0

$\Rightarrow$ f_1 is continuous at $f_2(x)$

$\Rightarrow$ $f = f_1 \circ f_2$ is continuous at any $x \in (3, \infty)$ #

② $f(x) = \dfrac{1}{5 - \sqrt{x^2+16}}$ is continuous at any $c \neq \pm 3$. because

$f = f_1 \circ f_2 \circ f_3$

where

continuous at any $c>0$

$f_1(x) = \dfrac{1}{5-x}$, $f_2(x) = \sqrt{x}$, $f_3(x) = x^2+16$

continuous at any $c \neq 5$

continuous at any c

$f_2 \circ f_3$ is continuous at any c.

$f = f_1 \circ f_2 \circ f_3$ is continuous if $f_2(f_3(x)) \neq 5 \Leftrightarrow x \neq \pm 3$

$f_2(f_3(x)) = \sqrt{x^2+16}$

#

③ $f(x) = \sqrt{1-x^2}$ is continous on $[-1, 1]$

because

$f = f_1 \circ f_2$, where

$f_1(x) = \underline{\sqrt{x}}$, $\quad f_2(x) = \underline{1 - x^2}$ ≥ 0 when $-1 \leq x \leq 1$

continuous on $[0, \infty)$

Application to limit

Thm

If $\lim\limits_{x \to c} f(x) = b$ and g is continuous
($x \to \infty$ or $x \to -\infty$)

at b, then

$$\lim_{\substack{x \to c \\ (x \to \infty \text{ or } x \to -\infty)}} g(f(x)) = g\Big(\lim_{\substack{x \to c \\ (x \to \infty \text{ or } x \to -\infty)}} f(x)\Big) = g(b)$$

Example

$$\lim_{x \to \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$$

because $\because \cos x$ is continuous

$$= \cos\left(\lim_{x \to \frac{\pi}{2}} \left(\underline{2x} + \sin\left(\frac{3\pi}{2} + x\right)\right)\right)$$

$2 \cdot \frac{\pi}{2} = \pi$

$$= \cos\left(\pi + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2}+x\right)\right)$$

$\because$ $\sin x$ is continuous

$\frac{3\pi}{2}+\frac{\pi}{2}=2\pi$

$$= \cos\left(\pi + \sin\left(\underbrace{\lim_{x\to\frac{\pi}{2}}\left(\frac{3\pi}{2}+x\right)}_{=2\pi}\right)\right)$$

$$= \cos(\pi + \sin 2\pi) = \cos\pi = -1 \quad \#$$

Application to solving equations

中間值定理

Thm (Intermediate Value Theorem, Thm 2.6.1)

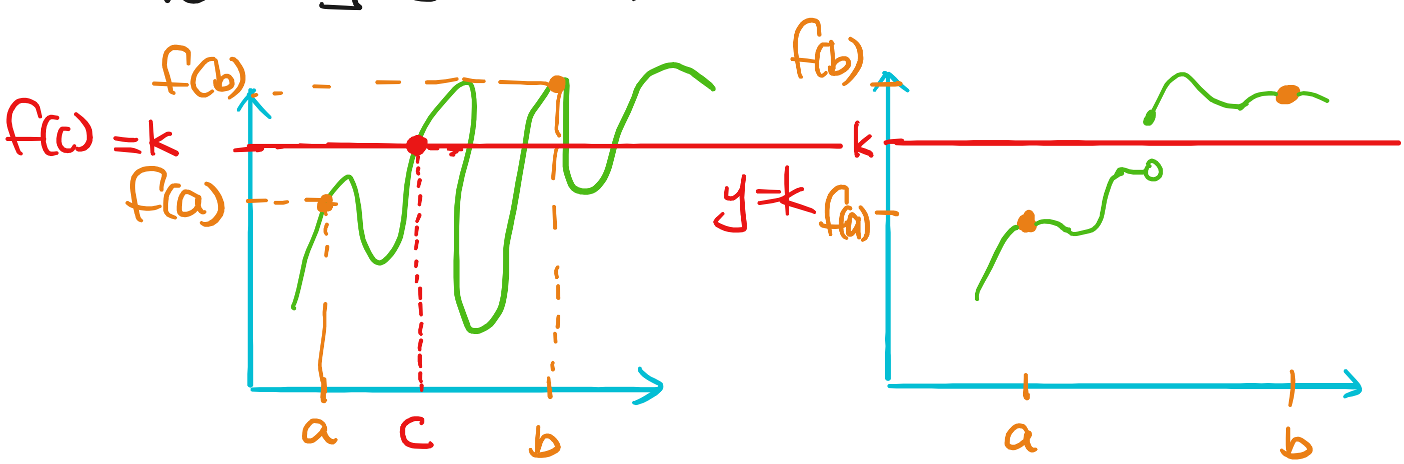
Let f be a continuous function on $[a,b]$. If $\exists\, k \in \mathbb{R}$ s.t. either

$$f(a) < k < f(b)$$

or

$$f(b) < k < f(a),$$

then $\exists\, c \in (a,b)$ s.t. $f(c) = k$.



$f(c)=k$

No c s.t.

$f(c)=k$

Example

Show that $x^4-x-1=0$ has a solution in $[-1,1]$.

pf

Note that

$$f(x)=x^4-x-1$$

is continuous on $[\underset{a}{-1}, \underset{b}{1}]$

Since

$$f(-1)=1>\underset{k}{0}$$

$$f(1)=-1<0$$

by the intermediate value thm,

$\exists\, c\in(-1,1)$ s.t. $f(c)=0$, where $f(c)=c^4-c-1$

$\Rightarrow c$ is a solution. #

Example

Show that $x^3-4x+2=0$ has 3

distinct roots in $[-3, 2]$.

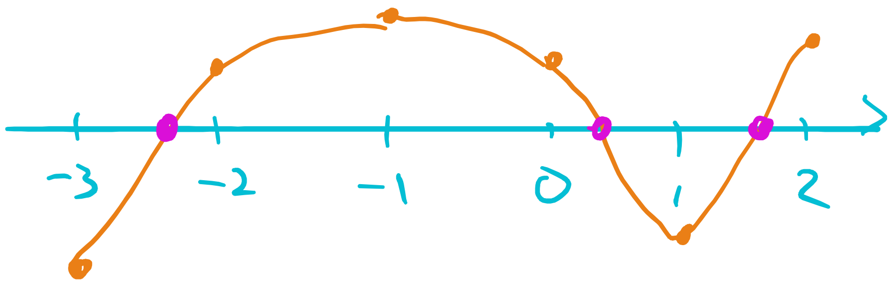
pf

Let $f(x) = x^3 - 4x + 2$.

Since

$f(-3) = -13$, $f(-2) = 2$, $f(-1) = 5$

$f(0) = 2$, $f(1) = -1$, $f(2) = 2$



by the intermediate value thm,

$f(x) = 0$ has a sol. in $(-3, -2)$,

a sol in $(0, 1)$ and a sol. in $(1, 2)$.

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Application to extreme values

Thm (Extreme Value Thm, Thm 2.6.2)

If f is continuous on $[a, b]$,

then f takes an (absolute) maximum

value M and an (absolute) minimum value m on $[a,b]$.

That is, $\exists\ c_1, c_2 \in [a,b]$ s.t.

$$f(c_2) = m \leq f(x) \leq M = f(c_1)$$

$\forall\ x \in [a,b]$. ↖ In particular, f is bounded on $[a,b]$
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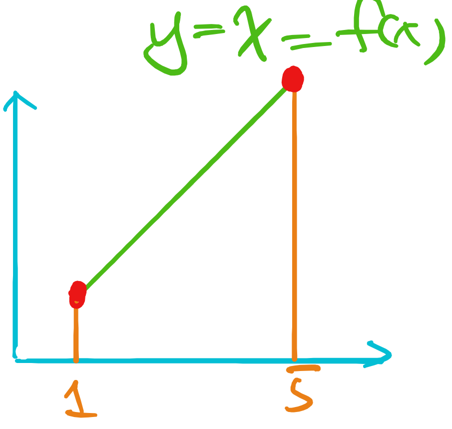
Example

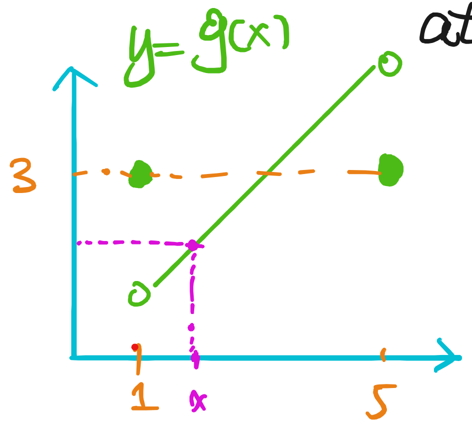
Consider two functions f, g on $[1,5]$:

① $f(x) = x$ is continuous on $[1,5]$.

And f attains a maximum value 5 at $x=5$

f '' minimum value 1 at $x=1$





② $g(x) = \begin{cases} 3, & x=1 \\ x, & 1<x<5 \end{cases}$ is NOT continuous

$$\begin{cases} \\ 3, & x=5 \end{cases} \quad \text{on } [1,5]$$