Calculus, week 11, Fall 24

§ The logarithm function

Recall:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

integer

$\forall n \neq -1.$

Q: $\int \frac{1}{x} \, dx = ?$

Def. (§ 7.2)

The function definition continuous on $[x, 1]$ or $[1, x]$ $0 \notin$

$$\boxed{\ln x := \int_1^x \frac{1}{t} \, dt}, \quad x > 0,$$

is called the (natural) logarithm function

Remark

Many people write "$\log x$" for "$\ln x$"

Thm (§7.3)

(i) The function $\ln x$ is a differentiable function from $(0,\infty)$ to $\mathbb{R}$

$$\ln : (0,\infty) \longrightarrow \mathbb{R}$$

(ii)

- $\frac{d}{dx}(\ln x) = \frac{1}{x} \to 0 \quad \forall x > 0.$

- $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \forall x \neq 0$

In particular, $\ln x$ is strictly increasing on $(0,\infty)$.

$\Rightarrow$ $\ln : (0,\infty) \to \mathbb{R}$

is one-to-one.

pf

$$\ln|x| = \begin{cases} \ln x & \text{if } x>0 \\ \ln(-x) & \text{if } x<0 \end{cases}$$

$$\Rightarrow \frac{d}{dx}(\ln|x|) = \begin{cases} \frac{d}{dx}(\ln x) = \frac{1}{x} & \text{if } x>0 \\ \frac{d}{dx}(\ln(-x)) & \text{if } x<0 \end{cases}$$

$$\overset{\text{chain rule}}{=} \frac{1}{-x} \cdot \frac{d(-x)}{dx} \qquad \frac{d(-x)}{dx} = -1$$

$$= \frac{1}{x}$$

#

NOTE:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

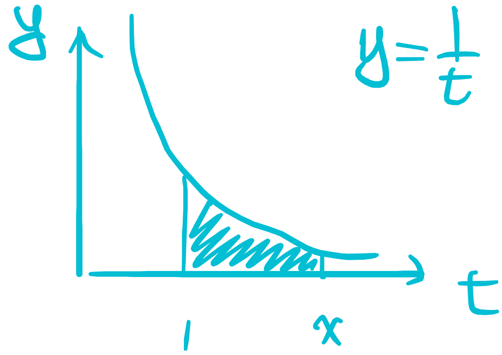
(iii) $\ln x = \int_1^x \frac{1}{t}\,dt$

$$\begin{cases} = \int_1^1 \frac{1}{t}\,dt = 0 & \text{if } x=1 \\ >0 & \text{if } x>1 \\ <0 & \text{if } x<1 \end{cases}$$

Remark

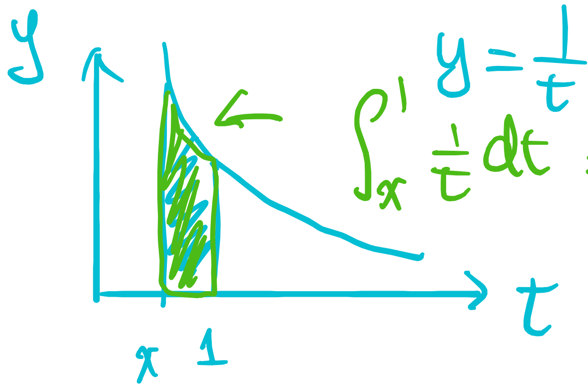
$$\ln x = \int_1^x \frac{1}{t}\,dt$$

$=$ the area of



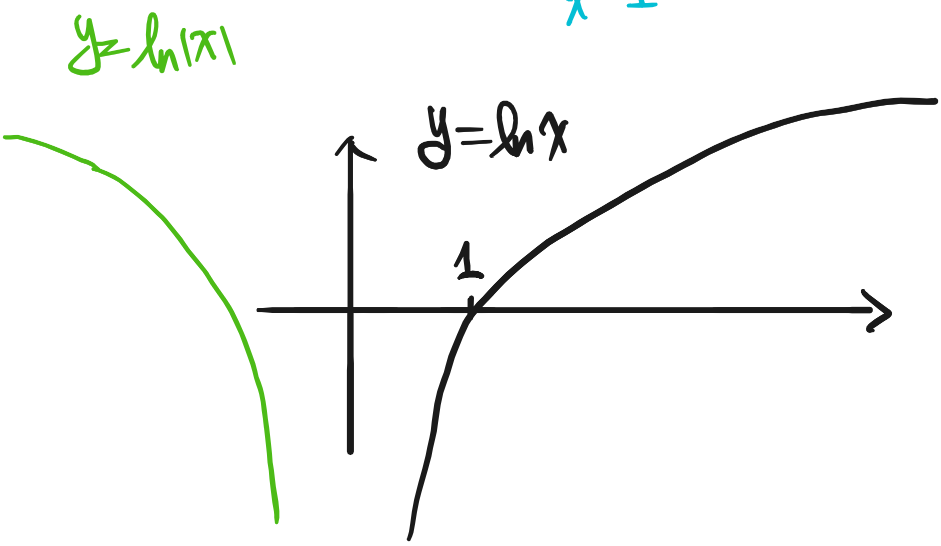
$(x>1)$

$= (-1)\cdot$ area of



$$\int_x^1 \frac{1}{t}\,dt = -\int_1^x \frac{1}{t}\,dt$$

$$= -\ln x$$



Thm(§7.2)

For any $a, b > 0$,

(i) $\ln(a \cdot b) = \ln a + \ln b$

(ii) $\ln\left(\frac{1}{b}\right) = -\ln b$

(iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

(iv) $\boxed{\ln\left(a^{\frac{p}{q}}\right) = \frac{p}{q} \cdot \ln a}$

pf

Since

$$\underline{\frac{d}{dx}\left(\ln(x \cdot b)\right)} \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{1}{x \cdot b} \cdot \underbrace{\frac{d(x \cdot b)}{dx}}_{= b}$$

$$= \frac{1}{x} = \underline{\frac{d}{dx}(\ln x)} \quad \forall x > 0$$

we have

$$\ln(x \cdot b) = \ln x + C.$$

for some constant C.

By taking $x=1$,

$$\ln b = \ln 1 + C = C$$

$$\Rightarrow \quad \ln(x \cdot b) = \ln x + \ln b$$

$$\Rightarrow \quad \ln(a \cdot b) = \ln a + \ln b \Rightarrow (i) \#$$

(ii) $$\ln\left(\frac{1}{b}\right) + \ln b \overset{(i)}{=} \ln\left(\frac{1}{b} \cdot b\right) = \ln 1 = 0$$

$$\Rightarrow \ln\left(\frac{1}{b}\right) = -\ln b. \quad \#$$

(iii) $$\ln\left(\frac{a}{b}\right) = \ln\left(a \cdot \frac{1}{b}\right) \overset{(i)}{=} \ln a + \ln \frac{1}{b}$$

$$\overset{(ii)}{=} \ln a - \ln b. \quad \#$$

(iv) ① $$\ln(a^p) = \ln(\overbrace{a \cdot (a \cdot \cdots \cdot a)}^{p \text{ times}})$$

$$\overset{(i)}{=} \ln a + \ln \underbrace{a^{p-1}}_{= a \cdot a^{p-2}} = \ln a + \ln a + \ln a^{p-2}$$

$$= \cdots = \overbrace{\ln a + \cdots + \ln a}^{p \text{ times}} = p \cdot \ln a.$$

② $\ln\left((a^{\frac{1}{q}})^{q}\right) = \ln a$

$\overset{①}{=}\ q\cdot\ln(a^{\frac{1}{q}})$ $\Rightarrow$ $\ln(a^{\frac{1}{q}}) = \frac{1}{q}\cdot\ln a.$

③ $\ln(a^{\frac{p}{q}}) = \ln\left((a^{\frac{1}{q}})^{p}\right)$

$\overset{①}{=} p\cdot\ln(a^{\frac{1}{q}}) \overset{②}{=} p\cdot\frac{1}{q}\cdot\ln a$

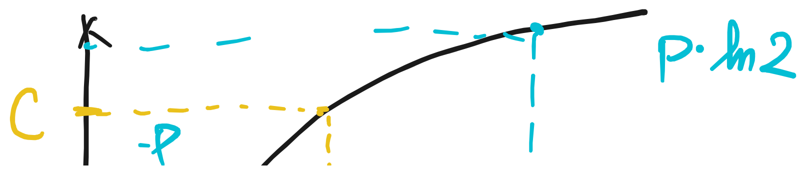
$= \frac{p}{q}\ln a.$ #

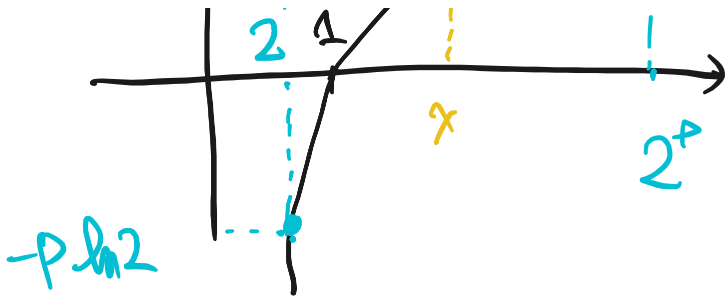
<u>NOTE</u> By (iv),

$\ln 2^{p} = p\cdot\overset{>0}{\ln 2} \longrightarrow +\infty$ as $p\to\infty$

$\ln \underset{\frac{1}{2^{p}}}{\underset{\shortparallel}{2^{-p}}} = -p\cdot\overset{>0}{\ln 2} \longrightarrow -\infty$ as $p\to\infty$

$y=\ln x$





$\forall c \in \ldots$ $\exists p$ s.t. $p \cdot \ln 2 > c$, $-p \cdot \ln 2 < c$

$$\ln : (0, \infty) \to (-\infty, \infty)$$

is onto $\Longleftarrow$ $\ln x$ is differentiable $\Rightarrow$ continuous

by IVT, $\exists x$ s.t. $\ln x = c$

Therefore,

$$\ln : (0, \infty) \to (-\infty, \infty) \qquad 1 \longmapsto 0, \quad e \longmapsto 1$$

is invertible, i.e. $\exists$

$$\exp : (-\infty, \infty) \to (0, \infty) \qquad 0 \longmapsto 1, \quad 1 \longmapsto e$$

s.t.

$$\exp(\ln x) = x \qquad \forall x > 0$$

$$\ln(\exp y) = y \qquad \forall y \in \mathbb{R}.$$

In particular, we have a number

$$\exp(1) \in \mathbb{R}$$

Notation:

$$e := \exp(1)$$

= the number with the property

$$\ln(e) = 1 = \int_1^e \frac{1}{t}\,dt$$

<u>Remark</u>

$$e = 2.718 \cdots$$

← "transcendental number"

That is, $\forall$ nonzero polynomial $p(x)$,

e is NOT a sol of $p(x) = 0$ $\rightsquigarrow$ $p(e) \neq 0$

$\Rightarrow$ e is an irrational number.

$2.718\cdots$

55

x	1	2	e	3	4
$\ln x$	0	0.69...	1	1.1...	1.39...

Example

① Find critical point(s) of

$$f(x) = x\ln x, \qquad x>0.$$

sol

Solve

$$f'(x) = 0$$

$$\begin{aligned} f'(x) &= (x)'\cdot \ln x + x\cdot(\ln x)' \\ &= \ln x + x\cdot\frac{1}{x} = \ln x + 1 \end{aligned}$$

$\Leftrightarrow$ $\ln x = -1$

$\ln e = 1$

$\Rightarrow \ln e^{p} = p\cdot \ln e$

$= p$

$\Rightarrow$ $x = e^{-1} = \frac{1}{e}$

is the critical point of $f(x)$. #

$= 2du$

② $\int^{2} (6x^2+2$

$$\int_1 \frac{\quad}{x^3+x+1}\,dx = ?$$

$\underbrace{x^3+x+1}_{= u}$

Sol

NOTE: $\left(\ln u(x)\right)' = \frac{1}{u(x)} \cdot u'(x)$

Let $u = x^3+x+1$

$\Rightarrow du = u' \cdot dx = (3x^2+1)\,dx$

So $\int_1^2 \frac{6x^2+2}{x^3+x+1}\,dx = \int_{u(1)}^{u(2)} \frac{2du}{u}$ ($u(2) = 11$, $u(1) = 3$)

$= 2 \cdot \ln|u| \Big|_{u=3}^{11}$

$= 2(\ln 11 - \ln 3)$

$= 2\ln\left(\frac{11}{3}\right)$ #

③ $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\,dx$ ($\sin x = (\cos x)'$, $\sin x\,dx = -du$)

Let $u = \cos x \Rightarrow du = -\sin x\, dx$

$$= \int -\frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

#

Double check:

$$(\ln|\sec x|)' \underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{1}{\sec x}\cdot(\sec x)'$$

$$= \cos x \cdot \tan x \sec x = \tan x.$$

Thm (Eq. (7.3.3))

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

§ Exponential function

Def.

The (natural) exponential function

$$\exp : (-\infty, \infty) \to (0, \infty)$$

is defined to be the inverse of

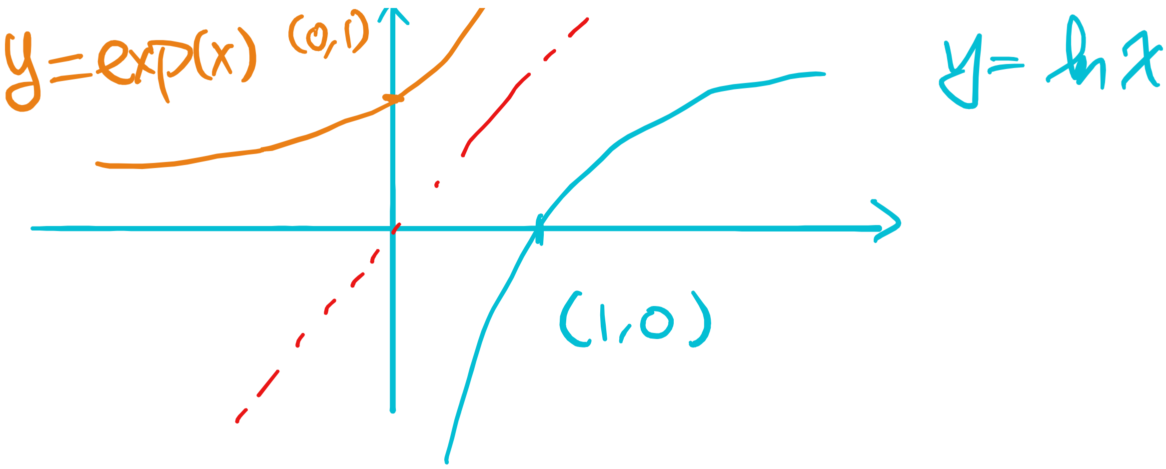
$$\ln : (0, \infty) \to (-\infty, \infty)$$

That is,

$$\ln(\exp(x)) = x \quad \forall\, x \in (-\infty, \infty)$$

$$\exp(\ln(y)) = y \quad \forall\, y \in (0, \infty).$$

$y = x$



<u>Thm</u> (§7.4)

(i) $\exp(0) = 1$

$\exp(1) = e$

(ii) $\exp(x) > 0 \quad \forall x \in \mathbb{R}$

(iii) $\lim\limits_{x \to -\infty} \exp(x) = 0$