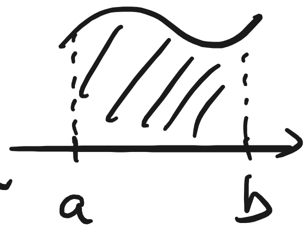
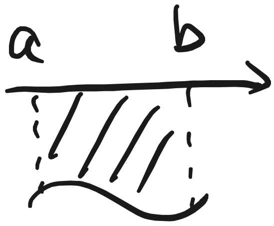
Calculus, week 10, Fall 24

<u>Recall</u>

① area (

$y=f(x)\geq 0$)

$$= \int_a^b f(x)\,dx$$

② area (

$y=g(x)\leq 0$)

$$= -\int_a^b g(x)\,dx$$

f, F: continuous on $[a,b]$

③ If $F'(x) = f(x) \quad \forall x \in (a,b)$,

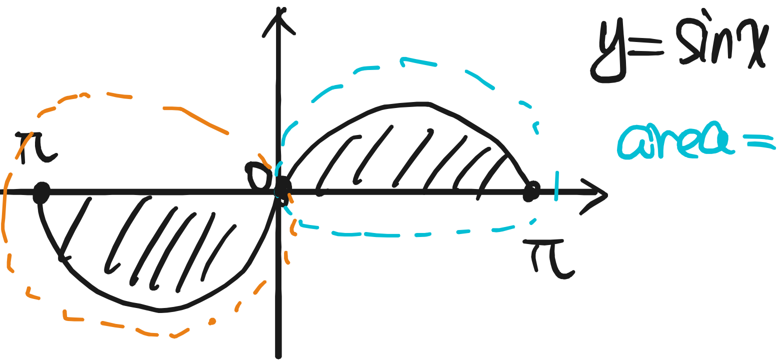
then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example

Find the area of the region



area $= -\int_{-\pi}^{0} \sin x\,dx$

area $= \int_0^{\pi} \sin x\,dx$

Sol

$$\text{area} = \int_0^{\pi} \overset{(-\cos x)'}{\overset{\shortparallel}{\sin x}}\,dx + \left(-\int_{-\pi}^{0} \sin x\,dx\right)$$

$$= -\cos x \Big|_0^{\pi} + (+1)\cdot(+\cos x)\Big|_{-\pi}^{0}$$

$$= -\overset{-1}{\underline{\cos \pi}} - (-\overset{1}{\underline{\cos 0}}) + \overset{1}{\underline{\cos 0}} - \overset{-1}{\underline{\cos(-\pi)}}$$

$$= 4 \qquad \#$$

§ Integration by substitution and Integration by parts

= u-subtitution

Thm (Integration by substitution, Thm 5.7.1 , Eq. 5.7.2)

If f is continuous, $F' = f$, u is differentiable, then

$$\boxed{\int_a^b f(u(x)) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du}$$

and

$$\int f(u(x))\, u'(x)\, dx = F(u(x)) + C$$

pf

By chain rule,

$$\frac{d}{dx}\left(F(u(x))\right) = F'(u(x)) \cdot u'(x)$$

$$\frac{d}{dx}\left(F(u(x))\right) = F'(u(x)) \cdot u'(x)$$

$$= f(u(x)) \cdot u'(x)$$

So
$$\int_a^b f(u(x)) \cdot u'(x)\, dx$$

$$= F(u(x)) \Big|_a^b = F(u(b)) - F(u(a))$$

$$= F(u) \Big|_{u=u(a)}^{u(b)}$$

$$= \int_{u(a)}^{u(b)} f(u)\, du \qquad \#$$

<u>Convenient notations:</u>

$$du = u'(x)\, dx \qquad \text{cf. } \frac{du}{dx} = u'(x)$$

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

<u>Example</u> <u>NOTE</u>

①

$$\int_0 \frac{1}{(3+5x)^2}\, dx = ?$$

$x^2 - (x)$

<u>Sol</u>

Let $u = 3+5x \Rightarrow du = (3+5x)' dx$

$= 5dx$

$$\Rightarrow \int_{x=0}^{x=1} \frac{1}{(\boxed{3+5x})^2} \boxed{dx} \quad \boxed{dx} = \frac{1}{5} du, \quad \boxed{3+5x} = u$$

$u(0) = 3$

$u(1) = 8$

$$= \int_{u(0)=3}^{u(1)=8} \underbrace{\frac{1}{u^2} \cdot \frac{1}{5}}_{=\frac{d}{du}\left(-\frac{1}{5}\frac{1}{u}\right)} du$$

$$= -\frac{1}{5}\frac{1}{u}\Big|_{u=3}^{8}$$

$$= -\frac{1}{40} + \frac{1}{15} = \frac{1}{24} \#$$

② $\int_0^1 x^2\sqrt{4+x^3}\, dx = ?$

$x^2 \, dx = \frac{1}{3} du$

<u>Sol</u>

Let $u = 4 + x^3 \Rightarrow du = 3x^2 dx$

So $\int_0^1 x^2\sqrt{4+x^3}\,dx$

$$= \int_{u(0)=4}^{u(1)=5} \sqrt{u}\;\frac{1}{3}\,du \qquad = \left(\frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\right)'$$

$$= \frac{2}{9}u^{\frac{3}{2}}\Big|_{u=4}^{5} = \frac{2}{9}\left(5\sqrt{5}-8\right)$$

#

③ $\int_0^2 (x^2-1)(x^3-3x+2)^3\,dx = ?$ $\quad = \frac{1}{3}du$

Sol

Let $u = x^3-3x+2 \Rightarrow du = (3x^2-3)\,dx$

So

$$= \int_{u(0)=2}^{u(2)=4} u^3\cdot\frac{1}{3}\,du = \frac{1}{3}\cdot\frac{1}{4}u^4\Big|_{u=2}^{4}$$

$$= 20$$

#

④ $\int_0^{\frac{1}{2}} \cos^3(\pi x)\cdot\sin(\pi x)\,dx = ?$

$$\int_0 \cos(\pi x) \; \underbrace{\sin(\pi x)\,dx}_{= -\frac{1}{\pi}du} = ?$$

Sol

Let $u = \cos(\pi x) \Rightarrow du = -\pi \sin(\pi x)\,dx$

So

$$= \int_{u(0)=1}^{u(\frac{1}{2})=0} u^3 \cdot \left(-\frac{1}{\pi}\right) du = -\frac{1}{\pi}\frac{u^4}{4}\Big|_{u=1}^{0}$$

$$= 0 - \left(-\frac{1}{\pi}\frac{1^4}{4}\right) = \frac{1}{4\pi} \quad \#$$

Thm (Integration by parts, §8.2) 分部積分

Suppose u, v are function with continuous derivatives. Then

$$\int u(x)\,v'(x)\,dx = u(x)v(x) - \int v(x)\,u'(x)\,dx$$

and

$$\boxed{\int_a^b u(x)\,v'(x)\,dx = u(x)\,v(x)\Big|_{x=a}^{b} - \int_a^b v(x)\,u'(x)\,dx}$$

$\int$

pf

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

$$\Rightarrow \int_a^b u(x)\cdot v'(x)\,dx = \underbrace{\boxed{\int_a^b (u(x)\cdot v(x))'\,dx}}_{=\, u(x)v(x)\Big|_a^b} - \int_a^b v(x)\cdot u'(x)\,dx$$

So

$$\int_a^b u(x)\cdot v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,u'(x)\,dx \quad \#$$

<u>Notations</u>

$$du = u'(x)\,dx$$

$$dv = v'(x)\,dx$$

$$\Rightarrow \int u\,dv = u\cdot v - \int v\,du$$

<u>Example</u>

① $\displaystyle\int_0^{\pi} \underbrace{x}_{u(x)}\,\overbrace{\boxed{\sin x}}^{v'(x)}\,dx = ?$

Sol

Let $u(x) = x \Rightarrow u'(x) = 1$

$v(x) = -\cos x \Rightarrow v'(x) = \sin x$

$$\Rightarrow \int_0^\pi x\sin x\,dx = \int_0^\pi u(x)\cdot v'(x)\,dx$$

$$= u\cdot v\Big|_{x=0}^{\pi} - \int_0^\pi v(x)\cdot u'(x)\,dx$$

$$= x\cdot(-\cos x)\Big|_0^\pi - \int_0^\pi (-\overset{(\sin x)'}{\boxed{\cos x}})\cdot 1\,dx$$

$$= -x\cos x\Big|_0^\pi + \sin x\Big|_0^\pi$$

$$= -\pi\cos\pi - (-0\cos 0) + \sin\pi - \sin 0$$

$$= \pi \quad \#$$

② $\displaystyle\int_0^1 \underbrace{x}_{u(x)}\,\overbrace{\sqrt{x+1}}^{v'(x)}\,dx = ?$

Sol

Let $\quad u(x) = x \quad \Rightarrow \quad u'(x) = 1$

$$v(x) = \frac{2}{3}(x+1)^{\frac{3}{2}} \quad \Rightarrow \quad v'(x) = \sqrt{x+1}$$

$$\Rightarrow \int_0^1 x\sqrt{x+1}\,dx = \int_0^1 u(x)\cdot v'(x)\,dx$$

$$= u(x)\,v(x)\Big|_0^1 - \int_0^1 v(x)\,u'(x)\,dx$$

$$= x\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^1 - \int_0^1 \underbrace{\frac{2}{3}(x+1)^{\frac{3}{2}}}_{\frac{d}{dx}}\cdot 1\,dx$$

$$= \frac{2}{3}\,2^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{\frac{5}{2}}\Big|_0^1$$

$$= \frac{\cancel{4}^{\,20}}{\cancel{3}_{\,15}}\sqrt{2} - \left(\underbrace{\frac{4}{15}\,2^{\frac{5}{2}}}_{\frac{16}{15}\sqrt{2}} - \frac{4}{15}\,1^{\frac{5}{2}}\right)$$

$$= \frac{4}{15}\sqrt{2} + \frac{4}{15} \quad \#$$

§ More properties of integrals

Thm (First mean value thm for integrals

Thm (First mean value thm for integrals
Thm 5.9.1)

If f is continuous on $[a,b]$
then $\exists\, c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

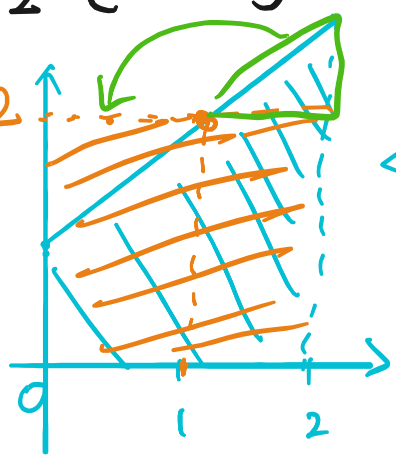
called the "average value" or "mean value" of f on $[a,b]$

Example

$$\int_0^2 \underbrace{x+1}_{}\!\!\overset{f(x)}{\,}\,dx = \frac{x^2}{2}+x\Big|_0^2 = \frac{4}{2}+2-0$$

$$=4$$

$$= 2\cdot(2-0) = f(\underset{=c}{1})\cdot(2-0)$$



$f(1)\cdot(2-1)$

$\int_0^2 f(x)\,dx$

__Thm__ (§5.8)

Suppose f and g are integrable

(i) $$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

(ii) If f is continuous on $[a,b]$,

then

$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x)) \cdot u'(x)$$

Pf

__Recall__:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Let $F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$

$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = \frac{d}{dx}\left(F(u(x))\right)$$

$$= F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x) \quad ☆$$

NOTE:

$$\because \frac{d}{dx}\left(\int_{u(x)}^{a} f(t)\,dt\right) = -\frac{d}{dx}\left(\int_{a}^{u(x)} f(t)\,dt\right)$$

$$= -f(u(x))\cdot u'(x)$$

$$\bullet\ \frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right) = \frac{d}{dx}\left(\int_{u(x)}^{a} f(t)\,dt + \int_{a}^{v(x)} f(t)\,dt\right)$$

$$= -f(u(x))\cdot u'(x) + f(v(x))\cdot v'(x)$$

(iii) If f is an odd function on $[-a,a]$

that is 奇函數

i.e. $f(-x) = -f(x)\ \forall x \in [-a,a]$

then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

pf

$$\int_{-a}^{a} f(x)\,dx = \boxed{\int_{-a}^{0} f(x)\,dx} + \int_{0}^{a} f(x)\,dx = 0$$

$y=-x$

$dy=-dx$

$$\int_{y(-a)=a}^{y(0)=0} f(-y)\,(-dy)$$

$$= \int_{a}^{0} f(y)\,dy$$

#

$$= -\cancel{\int_0^a} f(x)\,dx$$

(iv) If f is an even function on $[-a,a]$

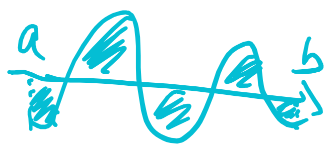
偶函數, i.e. $f(-x) = f(x)$ $\forall x \in [-a,a]$

then
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

(v) If f is continuous on $[a,b]$, $(a<b)$

and if
$$\int_a^b |f(x)|\,dx = 0$$

area of

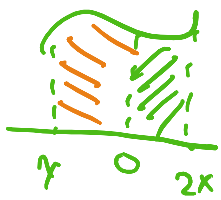


then
$$f(x) = 0 \quad \forall x \in [a,b]$$

Example

①
$$\frac{d}{dx}\left(\int_x^{2x} \frac{1}{1+t^2}\,dt\right)$$

$$||$$



$$\int_0^{2x} \frac{1}{1+t^2}\,dt + \int_x^{10} \frac{1}{1+t^2}\,dt$$

$$\parallel$$

$$\int_0^{2x} \frac{1}{1+t^2}\,dt - \int_0^{x} \frac{1}{1+t^2}\,dt$$

$$= \frac{d}{dx}\left(\int_0^{2x} \frac{1}{1+t^2}\,dt\right) - \frac{d}{dx}\left(\int_0^{x} \frac{1}{1+t^2}\,dt\right)$$

$$= \frac{1}{1+(2x)^2}\cdot(2x)' - \frac{1}{1+x^2}\cdot(x)'$$

$$= \frac{2}{1+4x^2} - \frac{1}{1+x^2} \qquad \#$$

② $$\int_{-1}^{1} \sin x\,dx = 0$$

$$\int_{-10}^{10} x^2 \sin x\,dx = 0$$

odd function:

$x, x^3, x^5, \dots$

$\sin x \leftarrow \sin(-x) = -\sin x$

$\sin x \cdot \cos x$

even function:

$1, x^2, x^4, x^6, \dots$

$\cos x, \ x\sin x$

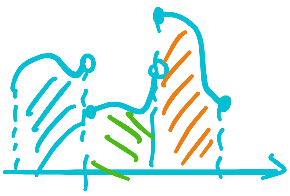
③ $$\int_{-1}^{1} x^2\,dx = 2\int_0^{1} x^2\,dx$$

$$= 2 \left. \frac{x^3}{3} \right|_0^1 = \frac{2}{3} \quad \#$$

Remark

Let

$$f(x) = \begin{cases} f_1(x), & a_0 \le x < a_1 \\ f_2(x), & a_1 \le x < a_2 \\ \vdots & \\ f_k(x), & a_{k-1} \le x \le a_k \end{cases}$$



and f_i are continuous on $[a_{i-1}, a_i]$

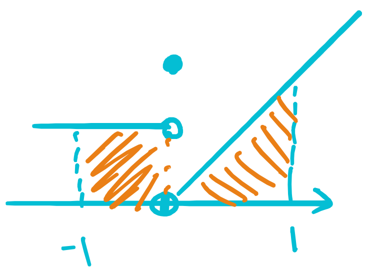
Then

$$\int_{a_0}^{a_k} f(x)\,dx = \int_{a_0}^{a_1} f_1(x)\,dx + \int_{a_1}^{a_2} f_2(x)\,dx + \cdots + \int_{a_{k-1}}^{a_k} f_k(x)\,dx$$

Example

If

$$f(x) = \begin{cases} x, & x > 0 \\ 2, & x = 0 \\ 1, & x < 0 \end{cases}$$



then

$$\int^1 f(x)\,dx = \int^0 \overbrace{f(x)}^{1 \,=}\,dx + \int^1 \overbrace{f(x)}^{x \,=}\,dx$$

$$= x\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = \frac{3}{2} \quad \#$$

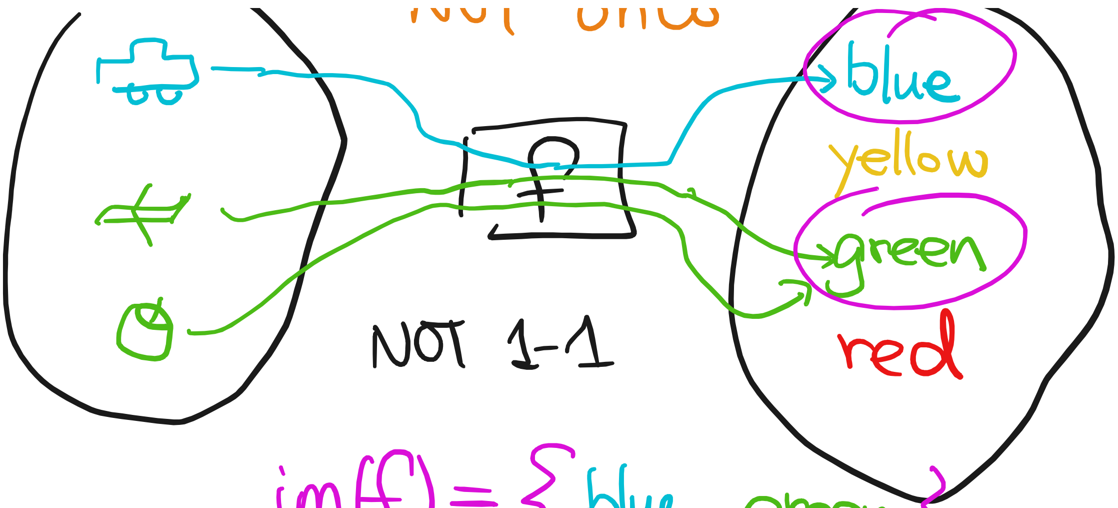
Transcendental functions

§ Inverse function

A function $f: A \to B$ has 3 ingredients:

- domain = 定義域 = A
  = the set of possible inputs
- codomain = 對應域 = B
  = the set of possible outputs
- how it works

e.g. A NOT B



$\mathrm{im}(f) = \{\text{blue}, \text{green}\}$

Def

Let $f: A \to B$ be a function

(i) We say f is one-to-one (or 1-1) if

"$x_1, x_2 \in A$, $x_1 \neq x_2$
$\Rightarrow f(x_1) \neq f(x_2)$"

(ii) The range 值域 (or image) of f

is the set

$$\mathrm{range}(f) = \mathrm{im}(f) = \{ f(x) \in B \mid x \overset{\text{in}}{\in} A \}$$

滿足 條件

(iii) We say f is onto if

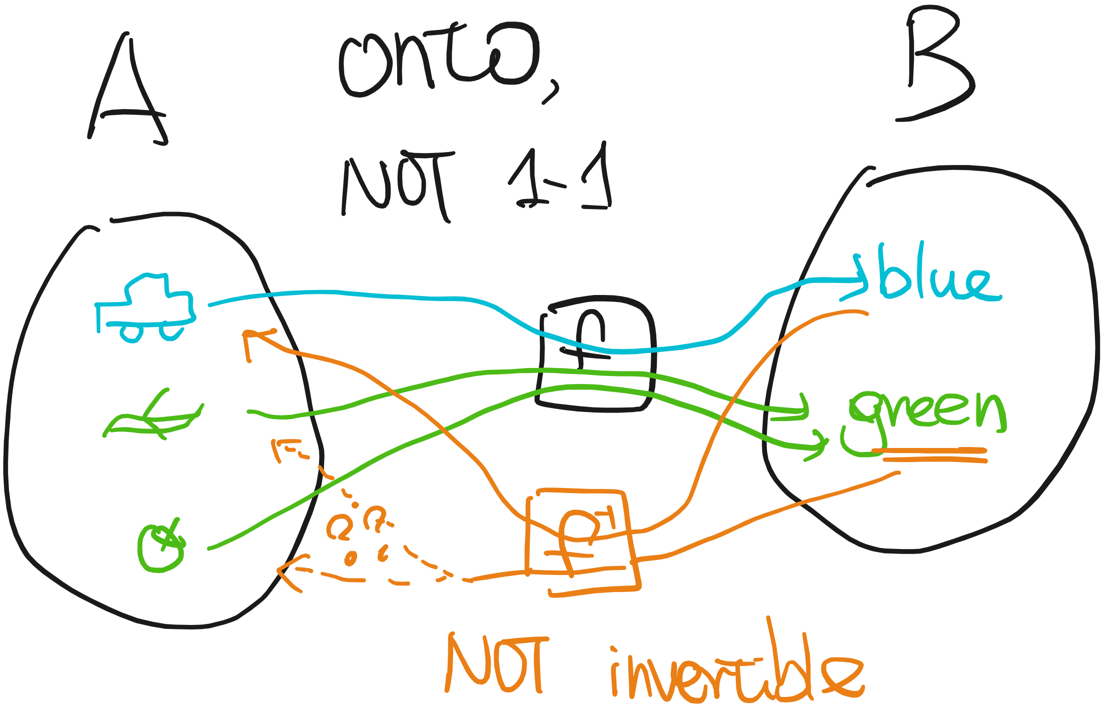
"$\forall y \in B, \exists x \in A$ s.t.

$f(x) = y$"

That is,

$$\mathrm{im}(f) = B$$

e.g.



Example

Let

- $f : (0,\infty) \longrightarrow (0,\infty)$,

  $f(x) = \dfrac{1}{x}$

- $g : (-\infty, 0) \cup (0,\infty) \to \mathbb{R}$

$$\breve{g}(x) = \frac{1}{x}$$

$\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

Then f is 1-1 and onto.

$(0,\infty)$

$\forall y \in (0,\infty), \exists x = \frac{1}{y}$ s.t. $f(\frac{1}{y}) = \frac{1}{\frac{1}{y}} = y$

g is 1-1, NOT onto

$\frac{1}{x} = g(x) \neq 0 \in \mathbb{R}$

$\forall x \in (-\infty, 0) \cup (0, \infty)$

Def

A function $f: A \to B$ is invertible 可逆

if $\exists\, g: B \to A$ s.t.

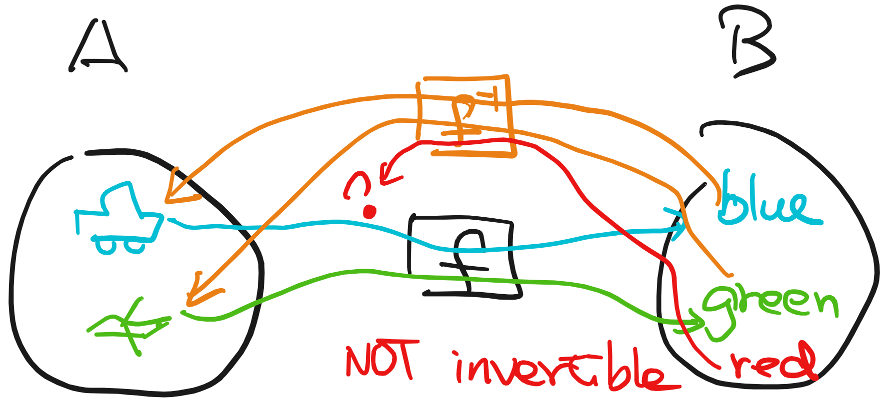
$$g(f(x)) = x \quad \forall x \in A$$

and

$$f(g(y)) = y \quad \forall y \in B$$

Such a function $g: B \to A$ is called the inverse of f, denoted by f^{-1}.

e.g.



Thm

A function $f: A \to B$ is invertible if and only if f is 1-1 and onto

Remark (Thm 7.1.2)

If $f: A \to B$ is 1-1, then the function

$$f: A \to \text{im}(f)$$

is 1-1 and onto, and hence invertible

Example

① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$

is 1-1 and onto.

To compute f^{-1}, solve x in

$$x^3 = f(x) = y$$

$$\Rightarrow \quad x = \sqrt[3]{y}$$

So $f^{-1}(y) = \sqrt[3]{y}$ #

② $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x-5$, is 1-1, onto

Solve $f(x) = 3x - 5 = y$

$$\Rightarrow \quad x = \frac{y+5}{3}$$

So $f^{-1}(y) = \dfrac{y+5}{3}$ #

③ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is NOT invertible

NOT 1-1:

$$-1 \neq 1$$

$$f(-1) = (-1)^2 = 1 = f(1)$$

NOT onto:

$$f(x) = x^2 \geq 0$$

so $f(x) = x^2 \neq -1 \in \mathbb{R} \quad \forall x \in \mathbb{R}$

Example

Show that $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^5 + 2x^3 + 3x - 4$$

is 1-1 " $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ "

or

pf

Method I : Check that " $\underbrace{f(x_1) = f(x_2)}$ $\Rightarrow x_1 = x_2$ "

(by definition)

" a problem of solving polynomial eq. of deg 5 "

Too difficult !!

Method II :

NOTE :

$$f'(x) = 5\underline{\underline{x^4}}_{\geq 0} + 6\underline{\underline{x^2}}_{\geq 0} + \underline{\underline{3}}_{> 0} > 0$$

$\forall x \in \mathbb{R}$.

$\Rightarrow f$ is strictly increasing on $\mathbb{R}$

So

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

So $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$$x_1 \neq x_2 \Rightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ \text{or} \\ x_2 < x_1 \Rightarrow f(x_2) < f(x_1) \end{cases}$$

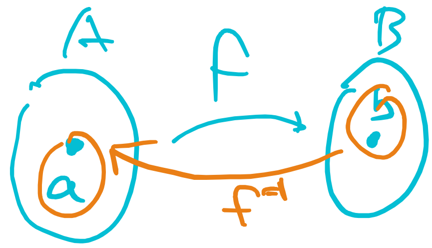
$$\Rightarrow f(x_1) \neq f(x_2)$$

So f is 1-1 on $\mathbb{R}$ #

<u>Thm</u> (Thm 7.1.8)

Let f be a 1-1 differentiable function and

$$f(a) = b.$$



If $f'(a) \neq 0$, then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

<u>pf</u>

By the definition of f^{-1},

$$f^{-1}(f(x)) = x$$

$\Rightarrow$

$$\frac{d}{dx}\left(f^{-1}(f(x))\right) = \frac{d}{dx}(x) = 1$$

$\|$ chain rule

$$(f^{-1})'(f(x)) \cdot f'(x)$$

$x=a$

$$\Rightarrow \quad (f^{-1})'(\underbrace{\boxed{f(a)}}_{=\,b}) \cdot f'(a) = 1$$

$$\Rightarrow \quad (f^{-1})'(b) = \frac{1}{f'(a)} \quad \#$$

<u>Example</u>

Let

$$f(x) = x^3 + \frac{x}{2}.$$

Find $(f^{-1})'(\underset{\uparrow b}{9})$.

<u>sol</u>

<u>NOTE</u>: ① $f'(x) = 3x^2 + \frac{1}{2} > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ f is 1-1

② It is easy to check that

$$f(\underset{\uparrow a}{2}) = 2^3 + \frac{2}{2} = 9$$

Thus,

$$(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{1}{3\cdot 2^2 + \frac{1}{2}}$$

$$= \frac{2}{25} \quad \#$$

HW6. 1:

Assume $\left| f(x) - (3x+2) \right| \leq |x|^{\frac{3}{2}}$

Show that (i) f is differentiable at 0

(ii) the tangent line of the graph of f at $(0, f(0))$ is $y = 3x+2$

pf

(i) ① NOTE: $\left| f(0) - (3\cdot 0 + 2) \right| \leq |0|^{\frac{3}{2}} = 0$

$$\Rightarrow \quad f(0) = 2$$

② $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$

know: $\left| f(h) - 3h - 2 \right| \leq |h|^{\frac{3}{2}}$

$= \lim \quad f(h) - 2 - 3h + 3h$

$$= \lim_{h\to 0} \frac{f(h) - 2 - 3h + 3h}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(h)-2-3h}{h} + 3 \right)$$

Since

$$0 \le \left| \frac{f(h)-2-3h}{h} \right| \le \frac{|h|^{\frac{3}{2}}}{|h|} = |h|^{\frac{1}{2}}$$

and

$$\lim_{h\to 0} |h|^{\frac{1}{2}} = 0,$$

by Pinching Thm,

$$\lim_{h\to 0} \frac{f(h)-2-3h}{h} \text{ exists, } = 0$$

So

$$f'(0) = \lim_{h\to 0} \left(\underbrace{\frac{f(h)-2-3h}{h}}_{\to 0} + 3 \right) \text{ exists,}$$

$$= 3$$

(ii) The tangent line at $(0, f(0))$
$= (0, 2)$

is

$$
\begin{aligned}
y-2 &= f'(0)\cdot(x-0) \\
&= 3\cdot(x-0)
\end{aligned}
$$

i.e.

$$y = 3x + 2 \qquad \#$$