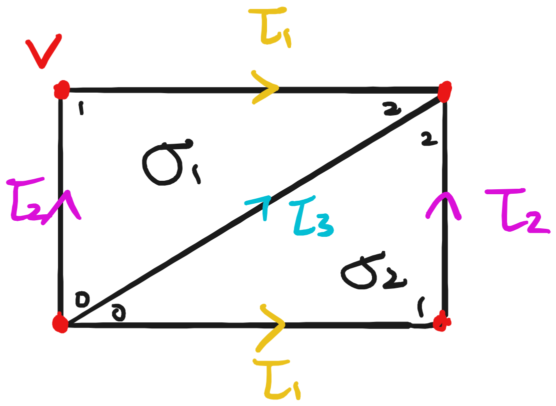
# Alg Topo 5/21

## Example

① Torus

$T =$ 


0-simplex: $V$

1-simplex: $\tau_1, \tau_2, \tau_3$

2-simplex: $\sigma_1, \sigma_2$

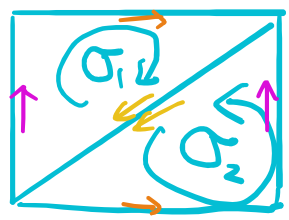
$$\partial(\sigma_1) = \sigma_1\big|_{[v_1,v_2]} - \sigma_1\big|_{[v_0,v_2]} + \sigma_1\big|_{[v_0,v_1]}$$

$$= \tau_1 - \tau_3 + \tau_2$$

$$\partial(\sigma_2) = \sigma_2\big|_{[v_1,v_2]} - \sigma_2\big|_{[v_0,v_2]} + \sigma_2\big|_{[v_0,v_1]}$$

$$= \tau_2 - \tau_3 + \tau_1$$

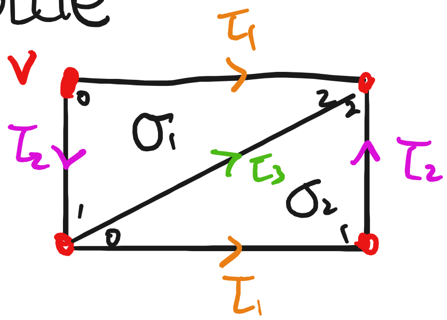
$$\Rightarrow \partial(\sigma_1 - \sigma_2) = 0$$




$$\Rightarrow \mu = [\sigma_1 - \sigma_2] \in H_2(T)$$

is a fundamental class. #

② Klein bottle

$K =$ 


$$\partial(\sigma_1) = \sigma_1|_{[1,2]} - \sigma_1|_{[0,2]} + \sigma_1|_{[0,1]}$$

$$= \tau_3 - \tau_1 + \tau_2$$

$$\partial(\sigma_2) = \tau_2 - \tau_3 + \tau_1$$

$\Rightarrow$

$$\partial(k_1\sigma_1 + k_2\sigma_2)$$

$$= k_1(\tau_3 - \tau_1 + \tau_2) + k_2(\tau_2 - \tau_3 + \tau_1)$$

$$= (k_1 + k_2)\tau_2 + (k_1 - k_2)(\tau_3 - \tau_1)$$

$$= 0 \quad \text{only if} \quad \begin{cases} k_1 + k_2 = 0 \\ k_1 - k_2 = 0 \end{cases}$$

$$\Rightarrow k_1 = k_2 = 0$$

So

$$\partial(k_1\sigma_1 + k_2\sigma_2) \neq 0 \text{ if } k_1, k_2 = \pm 1$$

$\Rightarrow$ NOT orientable #

## Euler characteristic

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank}(H_n(X))$$

is called the Euler characteristic of X.

**Thm**

Let X be a CW complex with finite cells, and

$$C_n = \# \text{ of } n\text{-cells in } X.$$

Then

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n C_n.$$

pf.

Let

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

be the cellular chain complex of X

$$\Rightarrow \quad H_n(X) = \ker(d_n) / \mathrm{im}(d_{n+1})$$

$$\Rightarrow \checkmark \operatorname{rank}(H_n(X)) = \operatorname{rank}(\ker(d_n)) - \operatorname{rank}(\mathrm{im}(d_{n+1}))$$

Applying the isomorphism thm to

$$d_n : C_n \to C_{n-1}$$

we have

$$\mathrm{im}(d_n) \cong C_n / \ker(d_n)$$

$$\Rightarrow \text{rank } \operatorname{im}(d_n) = \text{rank } C_n - \text{rank } \ker(d_n)$$

$$= c_n - \text{rank } \ker(d_n)$$

$$\Rightarrow \text{rank } H_n(X) = \text{rank } \ker(d_n) - \text{rank } \operatorname{im}(d_{n+1})$$

$$= \text{rank } \ker(d_n) - (c_{n+1} - \text{rank } \ker(d_{n+1}))$$

$$= \text{rank } \ker(d_n) + \text{rank } \ker(d_{n+1}) - c_{n+1}$$

Therefore,

$$\chi(X) = \text{rank } H_0(X) - \text{rank } H_1(X) + \text{rank } H_2(X) - \cdots$$

$$= (c_0 + \cancel{\text{rank } \ker(d_1)} - c_1)$$

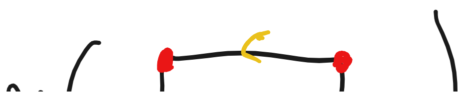
$$- (\cancel{\text{rank } \ker(d_1)} + \cancel{\text{rank } \ker(d_2)} - c_2)$$

$$+ \cdots$$

$$= c_0 - c_1 + c_2 - c_3 + \cdots$$

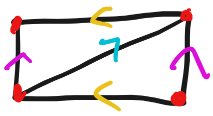
#

eg.



$\chi\Big($  $\Big) = 1 - 2 + 1 = 0$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$\chi\Big($  $\Big) = 1 - 3 + 2 = 0$

## Mayer – Vietoris sequence

**Prop** (p. 149)

Let $X$ be a space. $A, B \subset X$

$$X = \operatorname{int}(A) \cup \operatorname{int}(B)$$

Suppose

$$i : A \cap B \hookrightarrow A, \qquad j : A \cap B \hookrightarrow B$$

$$\alpha : A \hookrightarrow X, \qquad \beta : B \hookrightarrow X$$

are the inclusion maps.

Then the sequence

$$\cdots \to \underbrace{H_n(A \cap B)}_{x} \xrightarrow{\Phi} \underbrace{H_n(A)}_{y} \oplus \underbrace{H_n(B)}_{z} \xrightarrow{\Psi} H_n(X)$$

$$(*) \qquad \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

is exact, where

$$\Phi(x) = (i_* x \ , -j_* x)$$

$$\Psi(y,z) = \alpha_* y + \beta_* z$$

<u>Prop</u>

Suppose

$$X = A \cup B$$

with the property:

$$\exists\, U, V \subseteq_{\text{open}} X \quad \text{s.t.}$$

(i) $A \subset U, \quad B \subset V$

(ii) $U$ deformation retracts to $A$

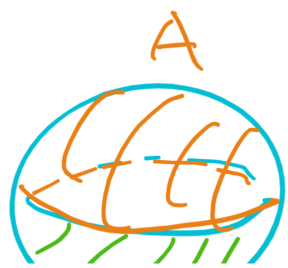
(iii) $V$ " $B$

(iv) $U \cap V$ " $A \cap B$

Then the seq. $\circledast$ is still exact.

<u>Example</u>

Consider $S^m \subseteq \mathbb{R}^{m+1}$



B

Take $A = \{x_{m+1} \geq 0\} \cap S^m$

$B = \{x_{m+1} \leq 0\} \cap S^m$

$\Rightarrow$

$$\cdots \to \underbrace{\tilde{H}_n(A) \oplus \tilde{H}_n(B)}_{0} \to \tilde{H}_n(S^m) \overset{\cong}{\to} \tilde{H}_{n-1}(\underbrace{A \cap B}_{\cong S^{m-1}}) \to \underbrace{\tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B)}_{0} \to \cdots$$

$$\Rightarrow \tilde{H}_n(S^m) \cong \tilde{H}_{n-1}(S^{m-1})$$

#

## Homology with coefficients

Let $G$ be an abelian group.

Define

$$C_n(X;G) = C_n(X) \otimes_{\mathbb{Z}} G$$

$$= \left\{ \sum m_\sigma \cdot \sigma \;\middle|\; \boxed{m_\sigma \in G},\ m_\sigma = 0 \text{ except finite } \sigma,\ \sigma : \Delta^n \to X \right\}$$

which is equipped with the boundary

homomorphism

$$\partial\left(\sum_\sigma m_\sigma \cdot \sigma\right) = \sum_\sigma \sum_{i=0}^{n} (-1)^i m_\sigma \cdot \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

$\Rightarrow \quad \partial^2 = 0$

The quotient group

$$H_n(X;G) = \ker\partial / \operatorname{im}\partial$$

is called the <u>n-th homology group with coefficients in G</u>

<u>Note</u>: $H_n(X) = H_n(X;\mathbb{Z})$.

## <u>Cohomology</u>

Let $X$ be a space, $G$ an abelian gp and $C_n(X) = \{$ singular n-chains in $X\}$

= free abelian group generated by

continuous maps $\sigma : \Delta^n \to X$

Define the group $C^n(X;G)$ of singular n-cochains with coefficients in G by

$$C^n(X;G) = \mathrm{Hom}(C_n(X), G)$$
$$= \left\{ \begin{array}{c} \text{all the group homomorphisms} \\ C_n(X) \longrightarrow G \end{array} \right\}$$

The coboundary homomorphism

$$\delta_n : \underbrace{C^n(X;G)}_{\varphi} \longrightarrow C^{n+1}(X;G)$$

is defined to be the dual of the boundary homomorphism

$$\partial_{n+1} : C_{n+1}(X) \longrightarrow C_n(X)$$

i.e.

$$\delta_n(\varphi) = \varphi \circ \partial_{n+1}$$

$$\begin{tikzcd} C_n(X) \arrow[r, "\varphi"] & G \arrow[d, equal] \\ C_{n+1}(X) \arrow[u, "\partial_{n+1}"] \arrow[r, "\delta_n(\varphi)"'] & G \end{tikzcd}$$

$$\delta_n(\varphi)(\sigma) = \varphi(\partial_{n+1}\sigma)$$

$\delta_n(\varphi)(\sigma) = \varphi(\partial_{n+1}\sigma)$ $\qquad \sigma \xrightarrow{\delta_n(\varphi)}$ $G$

$$= \sum_{i=0}^{n+1} (-1)^i \, \underline{\varphi(\sigma|_{[v_0 \cdots \hat{v}_i \cdots v_{n+1}]})}$$

## Lemma

$$\delta \circ \delta = 0 \qquad \delta_{n+1} \circ \delta_n = 0$$

$$C^n(X;G) \xrightarrow{\delta_n} C^{n+1}(X;G) \xrightarrow{\delta_{n+1}} C^{n+2}(X;G)$$

pf $(\delta \circ \delta)(\varphi) = \delta(\varphi \circ \partial) = \varphi \circ \boxed{\underline{(\partial \circ \partial)}}^{\,0} = 0$ #

So we have $\qquad$ cochain complex

$$0 \to C^0(X;G) \xrightarrow{\delta} C^1(X;G) \xrightarrow{\delta} \cdots \xrightarrow{\delta_{n-1}} C^n(X;G) \xrightarrow{\delta_n} C^{n+1}(X;G) \xrightarrow{\delta_{n+1}} \cdots$$

(**)

with the property

$$\operatorname{im}(\delta_{n-1}) \subset \ker(\delta_n)$$

The quotient group

$H^n(X;G) = \ker(\delta_n)/$

$$H(X;G) = \quad {}^{\phantom{x}}/_{\mathrm{im}(\delta_{n-1})}$$

is called the <u>n-th cohomology group</u> of X with coefficients in G.

Elements in $\ker(\delta)$ are called <u>cocycles</u>.

" $\mathrm{im}(\delta)$ " <u>coboundaries</u>

$(\ast\ast)$ is called the <u>singular cochain complex</u> of X with coefficients in G

One can get most of the properties of cohomology by dualizing the parallel properties of homology.

For example, if

$$f: X \to Y$$

is a continuous map, then one has

$$f_\# : C_n(X) \longrightarrow C_n(Y), \quad \sigma \mapsto f\circ\sigma$$

dualize $\rightsquigarrow$

$$f^\# : C^n(Y;G) \to C^n(X;G)$$

$$\varphi \longmapsto \varphi\circ f_\#$$

<u>Lemma</u>

$$\delta\circ f^\# = f^\#\circ\delta$$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] \arrow[d, "f_\#"'] \arrow[rrd, bend left, "\delta^X_n(f^\#(\varphi))"] \arrow[rrd, bend right=60, "f^\#(\delta_n(\varphi))"'] & C_n(X) \arrow[d, "f_\#"] \arrow[rd, "\varphi\circ f_\# = f^\#(\varphi)"] & \\
C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] \arrow[ur, phantom, "\circlearrowleft"] & C_n(Y) \arrow[r, "\varphi"'] & G
\end{tikzcd}$$

$$\partial^Y_{n+1}\circ f_\# = f_\#\circ\partial^X_{n+1}$$

$$(\varphi\circ \boxed{f_\#)\circ\partial^X_{n+1}} = \delta^X_n(\varphi\circ f_\#) = \delta^X_n(f^\#(\varphi))$$

$$\varphi\circ\partial^Y_{n+1}\circ f_\# = \delta^Y_n(\varphi)\circ f_\# = f^\#(\delta_n(\varphi))$$

By Lemma, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^n(Y;G) \arrow[r, "\delta"] \arrow[d, "f^\#"'] & C^{n+1}(Y;G) \arrow[r] \arrow[d, "f^\#"] & \cdots \\
\cdots \arrow[r] & C^n(X;G) \arrow[r] & C^{n+1}(X;G) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \to C^{n}(X;G) \xrightarrow{\delta} C^{n+1}(X;G) \to \cdots$$

Commutes

Therefore, we have the induced homomorphism

$$f^*: H^n(Y;G) \longrightarrow H^n(X;G) \qquad \forall n.$$
$$[\varphi] \longmapsto [f^\#(\varphi)]$$

The induced homomorphisms are functorial:

<u>Prop</u>

If $id_X: X \to X$ is the identity map, then

$$(id_X)^* = id_{H^n(X;G)} : H^n(X;G) \to H^n(X;G)$$

If $f: X \to Y$ and $g: Y \to Z$ are continuous maps, then

$$(g \circ f)^* = f^* \circ g^* : H^n(Z;G) \to H^n(X;G) \quad \forall n$$

See §3.1 for more properties of cohomology

## Cup product

Let $R$ be a ring.

For $\varphi \in C^p(X;R)$, $\psi \in C^q(X;R)$,

$\varphi : C_p(X) \to R$ $\qquad$ $\psi : C_q(X) \to R$

the cup product

$$\varphi \smile \psi \in C^{p+q}(X;R) \qquad C_{p+q}(X) \to R$$

is defined by $\quad \sigma : \Delta^{p+q} \to X$

$$(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\cdots,v_p]}\right) \cdot \psi\left(\sigma|_{[v_p,\cdots,v_{p+q}]}\right)$$

where

$$\sigma : \Delta^{p+q} = \left\{ (t_0,\cdots,t_{p+q}) \in \mathbb{R}^{p+q+1} \;\middle|\; \begin{array}{l} t_0+\cdots+t_{p+q}=1 \\ 1 \geq t_i \geq 0 \end{array} \right\} \to X$$

$$\sigma|_{[v_0,\cdots,v_p]} : \Delta^p \longrightarrow X$$

$[v_0, \cdots, v_p]$

$$\sigma|_{[v_0,\cdots,v_p]}(t_0,\cdots,t_p) = \sigma(t_0,\cdots,t_p,0,0,\cdots 0)$$

$$\sigma|_{[v_p,\cdots,v_{p+q}]} : \Delta^q \to X$$

$$\sigma|_{[v_p,\cdots,v_{p+q}]}(t_0,\cdots,t_q) = \sigma(0,\cdots,0,t_0,t_1,\cdots t_q)$$

**Lemma** (Lemma 3.6)

Let

$$C^\bullet(X;R) = \bigoplus_{n=0}^{\infty} C^n(X;R)$$

The pair $(C^\bullet(X;R), \cup)$ is an associative algebra with the property

$$\delta(\varphi \cup \psi) = \delta(\varphi) \cup \psi + (-1)^p \varphi \cup \delta(\psi)$$

$\forall\, \varphi \in C^p(X;R)$, $\psi \in C^\bullet(X;R)$.

This lemma guarantees $\exists$ well-defined

$$\cup : H^p(X;R) \times H^q(X;R) \to H^{p+q}(X;R)$$

$\cup : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$

because ... $\underset{\delta(\varphi)=0}{\overset{\varphi}{[\varphi]}}$ $\underset{\delta(\psi)=0}{\overset{\psi}{[\psi]}}$ $\longmapsto$ $\overset{\varphi}{[\varphi\cup\psi]}$

(i) If $\delta(\varphi)=0$ and $\delta(\psi)=0$, then

$$\delta(\varphi\cup\psi) = \underbrace{(\delta\varphi)}_{=0}\cup\psi \pm \varphi\cup\underbrace{\delta(\psi)}_{=0}$$

$$= 0+0 = 0$$

(ii) If $[\varphi]=[\varphi']$ and $[\psi]=[\psi']$

i.e. $\varphi' = \varphi + \delta\alpha$ and $\psi' = \psi + \delta\beta$

for some $\alpha, \beta$, then

$$\varphi'\cup\psi' = (\varphi+\delta\alpha)\cup(\psi+\delta\beta)$$

$$= \varphi\cup\psi + \underbrace{\boxed{\varphi\cup\delta\beta}}_{=\delta((-1)^p\cdot\varphi\cup\beta)} \quad + \delta\square$$

$$+ \underbrace{\boxed{\delta\alpha\cup\psi}}_{=\delta(\alpha\cup\psi)} + \underbrace{\boxed{\delta\alpha\cup\delta\beta}}_{=\delta(\alpha\cup\delta\beta)}$$

Note: $\delta(\alpha\cup\psi)$

$$\delta(\varphi\cup\beta) = \cancel{\underbrace{\delta\varphi}_{=0}\cup\beta} + (-1)^p\varphi\cup\delta\beta$$

$$= \varphi\cup\psi + \delta\left((-1)^p\varphi\cup\beta + \alpha\cup\psi + \alpha\cup\delta\beta\right)$$

$$\Rightarrow \quad [\varphi' \smile \psi'] = [\varphi \smile \psi]$$
in $H^{p+q}(X;R)$.

<u>Thm</u> (Thm 3.11)
Let $R$ be a commutative ring.
The cohomology

$$H^{\bullet}(X;R) = \bigoplus_{n=0}^{\infty} H^n(X;R),$$

equipped with the cup product, is a commutative graded ring, i.e. the cup product is an associative product on $H^{\bullet}(X;R)$ s.t. $\forall \alpha \in H^p(X;R)$, $\beta \in H^q(X;R)$,

$$\alpha \smile \beta \in H^{p+q}(X;R)$$

and

$$\alpha \smile \beta = (-1)^{pq} \beta \smile \alpha$$

<u>Prop</u>

If $f: X \to Y$ is a continuous map, then

$$f^*: H^\bullet(Y;R) \longrightarrow H^\bullet(X;R)$$

is a ring homomorphism:

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

$\forall\ \alpha, \beta \in H^\bullet(X;R)$.