Alg Topo 5/14

Orientation and Fundamental class

Def

A (topological) manifold of dimension n is a Hausdorff (and 2nd countable) topological space which is locally homeomorphic to $\mathbb{R}^n$.

A compact manifold is called a closed manifold

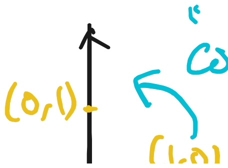
$\{(1,0),(0,1)\} \to \{(0,1),(1,0)\}$

Orientations for (real) vector spaces

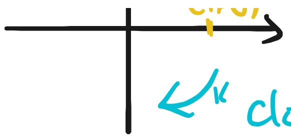
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\det = -1$

$\mathbb{R}^1$ "right"

$\mathbb{R}^2$ "counterclockwise"



"left"



"right-handed"

"left-handed"

<u>Def</u>

Let V be a finite-dimensional real vector space, and let

$$\mathcal{B}(V) = \text{the set of all the ordered bases for } V$$

We say that

$$(\vec{v}_1, \cdots, \vec{v}_n), \quad (\vec{w}_1, \cdots, \vec{w}_n) \in \mathcal{B}(V)$$

have the <u>same orientation</u> if the matrix of changing coordinates has a positive determinant, i.e. if

$$\vec{w}_i = a_{1i} \vec{v}_1 + a_{2i} \vec{v}_2 + \cdots + a_{ni} \vec{v}_n$$

$i=1,\dots,n$, then

$$\det(a_{ij}) > 0.$$

Having the same orientation is an equivalence relation $\sim$ with 2 equivalence classes.

An equivalence class in

$$\mathcal{B}(V)/\sim$$

is called an orientation of V

If an orientation is chosen, we say V is oriented.

We will identify an orientation

of V with a generator of

$$H_n(V, V \setminus \{0\}).$$

dimension of V

To do so, we need the five lemma:

<u>Five lemma</u> (p. 129)

Suppose we are given a commutative diagram

exact

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

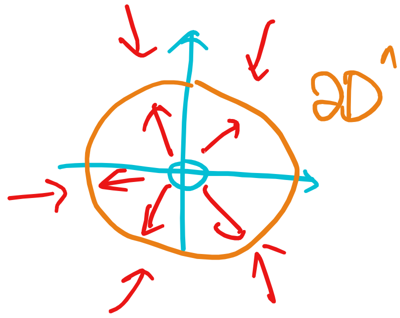
whose two rows are exact.

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Let $V = \mathbb{R}^n$. Since

$$D^n \hookrightarrow V$$

$$\partial D^n \hookrightarrow V \setminus \{0\}$$



induce iso

$$H_k(D^n) \xrightarrow[i_*]{\cong} H_k(V)$$

$$H_k(\partial D^n) \xrightarrow[j_*]{\cong} H_k(V \setminus \{0\}),$$

the map

$$(D^n, \partial D^n) \hookrightarrow (V, V \setminus \{0\})$$

also induces iso

$$H_k(D^n, \partial D^n) \xrightarrow[\cong]{k_*} H_k(V, V \setminus \{0\})$$

by Five lemma:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(\partial D^n) \arrow[r] \arrow[d, "j_*"', "\cong"] & H_k(D^n) \arrow[r] \arrow[d, "i_*"', "\cong"] & H_k(D^n, \partial D^n) \arrow[r] \arrow[d, "\cong"', "k_*"] & H_{k-1}(\partial D^n) \arrow[r] \arrow[d, "j_*"', "\cong"] & H_{k-1}(D^n) \arrow[r] \arrow[d, "i_*"', "\cong"] & \cdots \\
\cdots \arrow[r] & H_k(V \setminus \{0\}) \arrow[r] & H_k(V) \arrow[r] & H_k(V, V \setminus \{0\}) \arrow[r] & H_{k-1}(V \setminus \{0\}) \arrow[r] & H_{k-1}(V) &
\end{tikzcd}$$

$\cdots \to H_k(V \setminus \{0\}) \to H_k(V) \to H_k(V, V\setminus\{0\}) \to H_{k-1}(V\setminus\{0\}) \to H_{k-1}(V) \cdots$

<u>Prop</u>

Let V be a real vector space of dim. n. The orientations of V are naturally in one-in-one correspondence to the generators of

$$H_n(V, V \setminus \{0\}) \cong \mathbb{Z}.$$

<u>pf</u>

Recall that a homeomorphism

$$\begin{array}{ccc} \Delta^n & \longrightarrow & D^n \\ \shortparallel & & \\ \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1,\ t_0, \cdots, t_n \geq 0\} & & \end{array}$$

induces a generator of

$$H_n(D^n, \partial D^n) \xrightarrow[k_*]{\cong} H_n(V, V\setminus\{0\})$$

Given a basis

Given a basis

$$\beta = (\vec{v}_1, \dots, \vec{v}_n) \in \mathcal{B}(V),$$

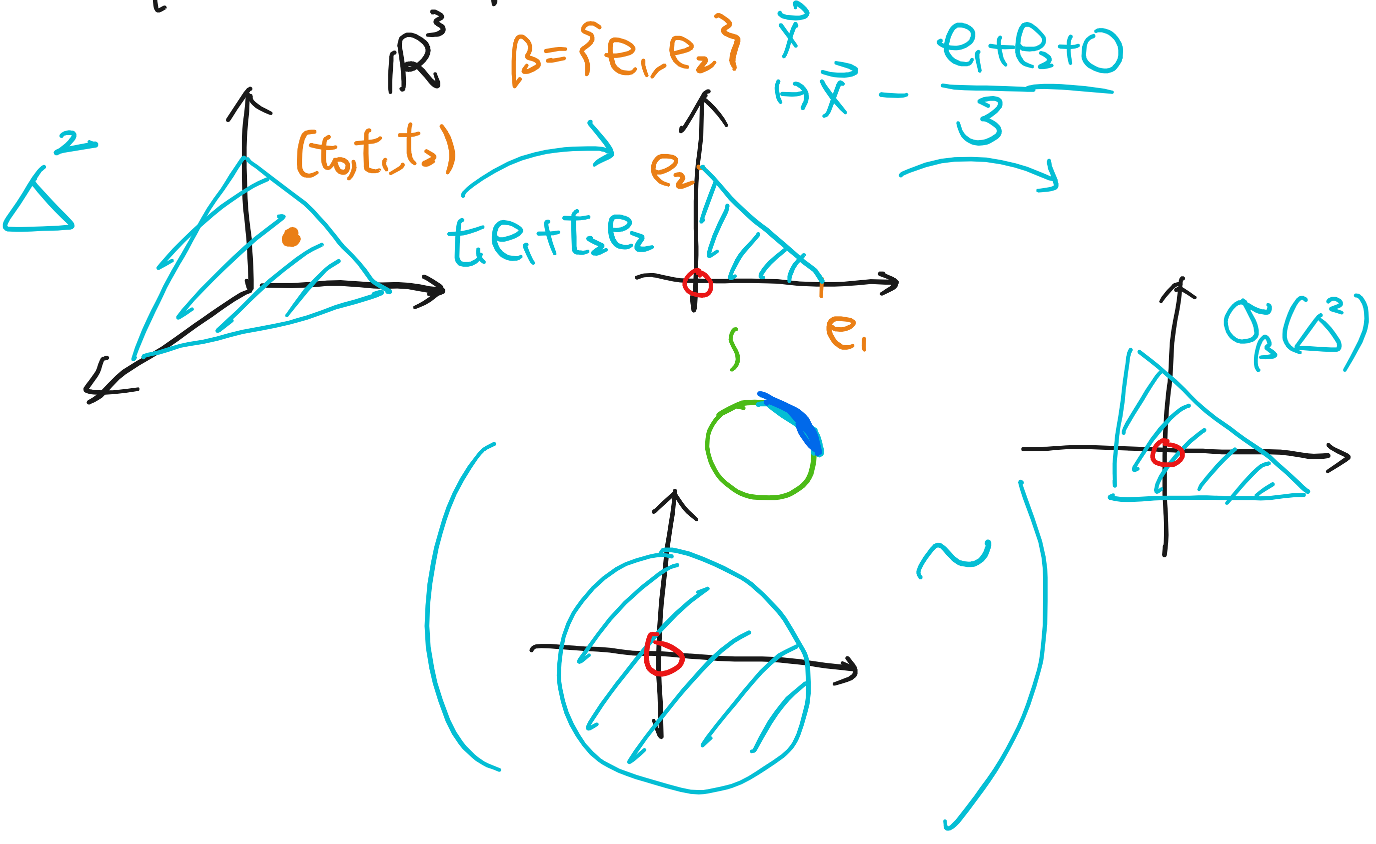
the map

$$\sigma_\beta : \Delta^n \longrightarrow V$$

$$\sigma_\beta(t_0, \dots, t_n) = \sum_{i=1}^{n} t_i \vec{v}_i - \frac{1}{n+1} \sum_{i=1}^{n} \vec{v}_i$$

defines a homeomorphism

$$\sigma_\beta : \Delta^n \to \sigma_\beta(\Delta^n) \cong D^n$$



So we have a generator

$\cup$

$$[\sigma_\beta] \in H_n(\sigma_\beta(\Delta^n), \sigma_\beta(\Delta^n) \setminus \{0\})$$

$$\downarrow \text{ } \wr\!\wr \text{ (five lemma)}$$

$$H_n(V, V \setminus \{0\})$$

So we have a map

$$\mathcal{B}(V) \longrightarrow \left\{ \begin{array}{l} \text{generators} \\ \text{of } H_n(V, V\setminus\{0\}) \end{array} \right\}$$

To objection between orientations and generators, it suffices to show

$$\beta \sim \gamma \iff [\sigma_\beta] = [\sigma_\gamma]$$

in $H_n(V, V \setminus \{0\})$

Observe that if

$$\gamma = \{ w_1, \cdots, w_n \}$$

is another basis s.t.

$$\vec{w}_i = a_{1i} \vec{v}_1 + \cdots + a_{ni} \vec{v}_n ,$$

$i = 1, \cdots, n$, then the map

$$A \; \cancel{\Phi} : V \to V,$$

$$A \; \cancel{\Phi}\Big(\underbrace{\sum_{i=1}^{n} x_i \vec{v}_i}_{\beta} \Big) = \underbrace{\sum_{i=1}^{n} x_i \vec{w}_i}_{\gamma}$$

$$= \sum_{i,j=1}^{n} \underbrace{a_{ij} x_j}_{A \cdot \vec{x}} \vec{v}_i$$

$A = (a_{ij})$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

maps $\sigma_\beta(\Delta^n)$ to $\sigma_\gamma(\Delta^n)$

$\Rightarrow$

$$A_* : H_n(V, V - \{0\}) \to H_n(V, V - \{0\})$$

$$A_*([\sigma_\beta]) = [\sigma_\gamma]$$

So

$$[\sigma_\beta] = [\sigma_\gamma]$$

$$\Longleftrightarrow \quad \deg(A) = 1$$

Claim:

$$\deg(A) = \operatorname{sign}(\det(A)) \quad (*)$$

idea

By elementary row operations,

$$A = E_1 E_2 \cdots E_l$$

where E_i are one of the following types:

$$S_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}, \quad E_{ij\lambda} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \lambda & \ddots & \\ & & & & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & \\ & \ddots \end{pmatrix}$$

$$M_{i;c} = \begin{pmatrix} c & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad c \neq 0$$

Check all of them satisfy ⊛.

$$\deg(A) = \operatorname{sign}(\det(A)). \qquad \#$$

Let M be a manifold, $x \in M$.

$\exists$ neighborhood V of x s.t.

$V \cong \mathbb{R}^n$. By the excision thm,

$(x \leftrightarrow 0)$

$$\begin{array}{c} H_n(V, V \setminus \{x\}) \cong H_n(M, M \setminus \{x\}) \\ \wr\wr \\ H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{array}$$

Def $(\dim M = n)$

A local orientation of M at $x \in M$

is a generator μ_x of $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$.

An orientation of M is a function

$$\begin{aligned} M &\longrightarrow \coprod_{x \in M} H_n(M, M \setminus \{x\}) \\ x &\longmapsto \mu_x \in H_n(M, M \setminus \{x\}) \end{aligned}$$

local orientation

satisfying a certain compatibility condition.

If an orientation exists for M, we say M is orientable

Thm (Thm 3.26)

Let M be a closed connected mfd of dim n.

(i) If M is orientable, then

$$\boxed{H_n(M) \longrightarrow H_n(M, M \setminus \{x\})}$$

see long exact seq of $(M, M \setminus \{x\})$

is an isomorphism for each $x \in M$.

(iii) $H_k(M) = 0 \quad \forall\, k > n$.

Def

Let M be a manifold (not necessarily closed connected). An element of $H_n(M)$ whose image in

$$\mu_x \in H_n(M, M \setminus \{x\})$$

is a generator for each $x \in M$ is called a fundamental class or an orientation class for M.

Thm (p. 236)

A fundamental class exists iff M is closed and orientable.

Construction of fundamental class

Recall Δ-complex

Prop

The cellular chain complex of a Δ-complex is isomorphic to its simplicial chain complex.
In particular, the simplicial homology groups of a Δ-complex are isomorphic its singular homology groups.

Suppose a closed manifold M of dim n has a Δ-complex structure on it.

Let

$\underbrace{\sigma_1, \cdots, \sigma_m}$: $\Delta^n \to M$ ↙ generators of n-chains (simplicial)

be all the n-simplices of M.

A homology class $\mu \in H_n(M)$ is represented by

$$\mu = \left[\sum_{i=1}^{m} k_i \sigma_i \right]$$

for some $k_i \in \mathbb{Z}$.

<u>Prop</u> (p. 238)

A closed n-manifold M is orientable iff $\exists\, k_i = \pm 1$ s.t.

$$\partial \left(\sum_{i=1}^{m} k_i \sigma_i \right) = 0.$$

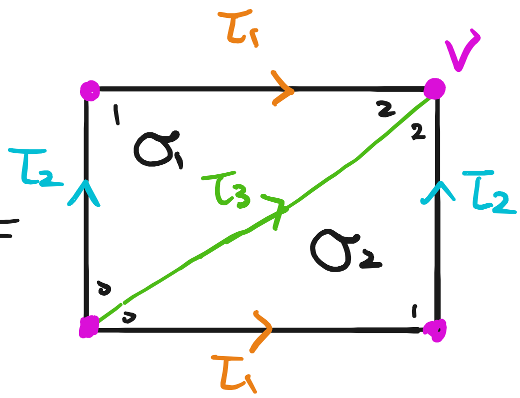
In this case, the class

$$\mu = \left[\sum_{i=1} k_i \sigma_i \right] \in H_n(M)$$

is a fundamental class for M.

Example

① Torus $\quad T =$



Note:

$$\partial(\sigma_1) = \sigma_1|_{[v_1,v_2]} - \sigma_1|_{[v_0,v_2]} + \sigma_1|_{[v_0,v_1]}$$

$$= \tau_1 - \tau_3 + \tau_2$$

$$\partial(\sigma_2) = \sigma_2|_{[v_1,v_2]} - \sigma_2|_{[v_0,v_2]} + \sigma_2|_{[v_0,v_1]}$$

$$= \tau_2 - \tau_3 + \tau_1$$

$$\Rightarrow \quad \partial(\sigma_1 - \sigma_2) = 0$$

$$\Rightarrow \quad \mu = [\sigma_1 - \sigma_2] \in H_2(T)$$

$\mu = [\sigma_1 - \sigma_2] \in H_2(T)$

is a fundamental class of T #