Recall

The singular homology groups of a CW cx X can be computed by

$$\cdots \to H_n(X^n, X^{n-1}) \xrightarrow{d_n = d} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

where

$$H_n(X^n, X^{n-1}) = \text{free abelian gp with basis } e_\alpha^n : \text{n-cells of } X$$

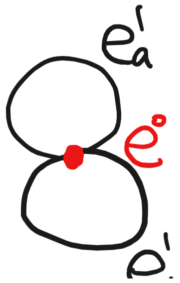
and

$$d(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1}.$$

$d_{\alpha\beta}$ is the degree of

$$S^{n-1} = \partial D_\alpha^n \xrightarrow[\text{map}]{\text{attaching}} X^{n-1} \longrightarrow X^{n-1} / (X^{n-1} - e_\beta^{n-1}) \cong S^{n-1}$$

Example

$X =$ 

$X_0 = \bullet$

$X_1 = X$

$\smile_b$

Cellular complex:

$$\begin{array}{ccccccc} {\scriptstyle 2} & & {\scriptstyle 1} & & {\scriptstyle 0} & & \\ 0 & \to & \mathbb{Z}e^1_a \oplus \mathbb{Z}e^1_b & \xrightarrow{d \,=\, 0} & \mathbb{Z}e^0 & \to & 0 \end{array}$$

$$e^1_a \longmapsto e^0 - e^0 = 0$$

$$e^1_b \longmapsto e^0 - e^0 = 0$$

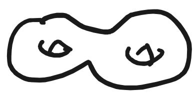
$\Rightarrow$

$$H_0(X) \cong \mathbb{Z}/0 \cong \mathbb{Z}$$

$$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$H_k(X) = 0 \quad \forall k \neq 0, 1.$$

#

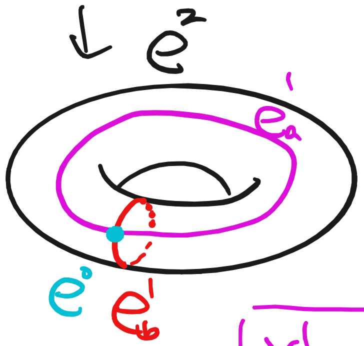


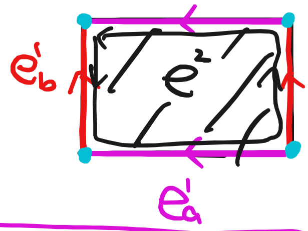
genus = 2 ...

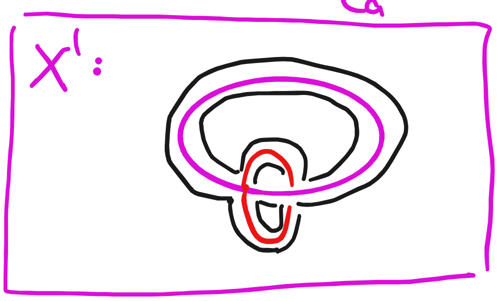
genus = 1

<u>Example</u>

$T = S^1 \times S^1$







$$= e^0 \cup \underline{e^1_a \cup e^1_b} \cup e^2$$

Cellular cx:

$$\begin{array}{cccc} {\scriptstyle 3} & {\scriptstyle 2} & {\scriptstyle 1} & {\scriptstyle 0} \end{array}$$

$0 =$

$$0 \to \mathbb{Z}e^2 \xrightarrow{d_2} \mathbb{Z}e^1_a \oplus \mathbb{Z}e^1_b \xrightarrow{d_1} \mathbb{Z}e^0 \to 0$$

$$d_2(e^2) = d_a \cdot e^1_a + d_b e^1_b$$

$$d_a = \deg\left(S^1 = \partial D^2 \longrightarrow X^1 \longrightarrow X^1/X^1 - e_a \cong S^1\right)$$

$$= 0$$

Similarly, $d_b = 0$

$$\Rightarrow d_2 = 0$$

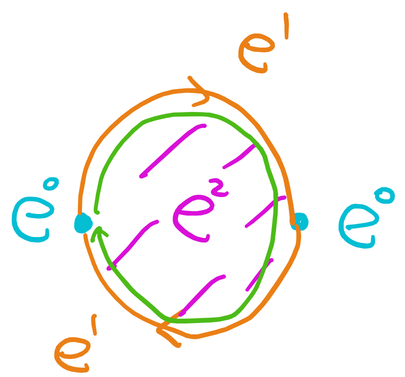
So

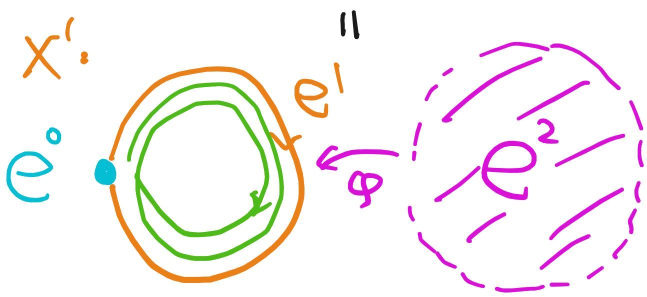
$$H_n(T) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{other} \end{cases}$$

#

<u>Example</u>

$$X = \mathbb{R}P^2 = e^0 \cup e^1 \cup_\varphi e^2$$





$$\deg\left(S^1 = \partial D \xrightarrow{\varphi} X^1 \to X^1/_{X^1 - e^1} \cong S^1\right) = 2$$

Cellular cx:

$$0 \to \mathbb{Z}e^2 \xrightarrow{\times 2} \mathbb{Z}e^1 \xrightarrow{0} \mathbb{Z}e^0 \to 0$$

$$e^2 \longmapsto 2e^1$$

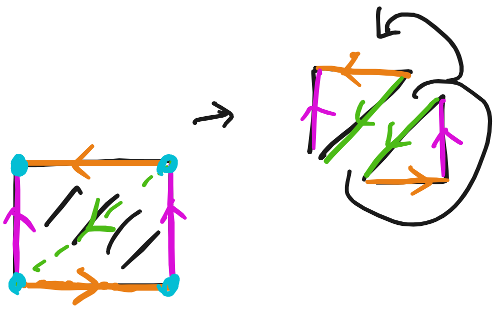
So $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$

$H_1(\mathbb{R}P^2) \cong \mathbb{Z}/_{2\mathbb{Z}}$

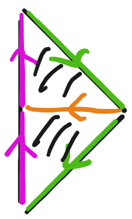
$H_k(\mathbb{R}P^2) = 0 \quad \forall k \neq 0,1$ $\#$

<u>Example</u>

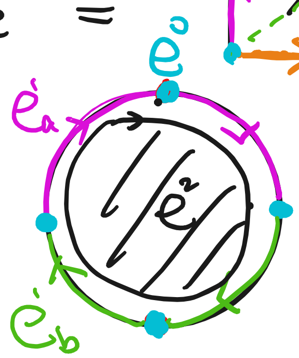
K = Klein bottle =



$\cong$

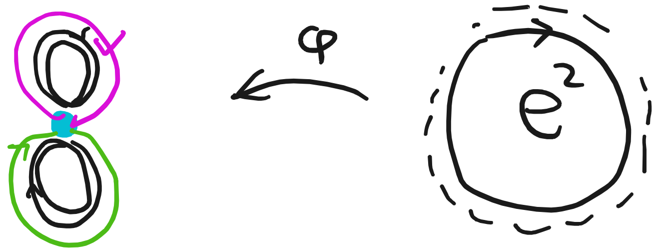


$\cong$



$\cong e^0 \cup e^1_a \cup e^1_b \cup_\varphi e^2$

K^1:



$\Rightarrow$

$$\deg\left(S^1 = \partial D^2 \to K^1 \to K^1/(K^1 - e_a^1) \cong S^1\right) = 2$$

$$\deg\left(\quad \text{''} \quad e_b^1 \quad \right) = 2$$

Cellular cx:

$$0 \to \mathbb{Z}e^2 \to \mathbb{Z}e_a^1 \oplus \mathbb{Z}e_b^1 \xrightarrow{0} \mathbb{Z}e^0 \to 0$$

$$e^2 \longmapsto 2e_a^1 + 2e_b^1$$

$(\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z})$

So

$H_0(K) \cong \mathbb{Z}$

$[(x,y)] \longmapsto (x-y, [y])$

$H_1(K) \cong \mathbb{Z}\oplus\mathbb{Z}/\langle(2,2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$

$H_n(K) = 0 \quad \forall\, n \neq 0, 1$ #

Consider

$$\mathbb{Z}\oplus\mathbb{Z} \underset{\text{onto}}{\xrightarrow{\phi}} \mathbb{Z}\oplus\mathbb{Z}_2$$

$$(x, y) \longmapsto (x-y, [y])$$

$$\ker(\phi) = \left\{ (x,y) \;\middle|\; \begin{array}{l} x-y=0 \\ y=2k \end{array} \right\}$$

$$= \{(2k, 2k)\} = \langle (2,2) \rangle$$

By iso thm.

<u>exer</u>: ↙ search "classification of surfaces"

① Study the classification of closed surfaces

② Compute the homology groups of each surface.

see p.5 and Example 2.37.

<u>Next goal</u>: Compute $H_k(\mathbb{RP}^n)$

§ Computation of degrees

<u>Key</u>:

- local degree
- generator of $H_n(S^n)$

Local degree

Let

$$f : S^n \longrightarrow S^n, \qquad n > 0.$$

Suppose $\exists\, y \in S^n$ s.t.

$$f^{-1}(y) = \{x_1, \cdots, x_m\}$$

is a finite set.

Let $U_1, \cdots, U_m$ be disjoint neighborhoods of $x_1, \cdots, x_m$ and V be a neighborhood of y

of $x_1, \dots, x_m$

s.t.

$$f(U_i) \subseteq V \qquad \text{for } i = 1, \cdots, m$$

Then

$$f(U_i - \{x_i\}) \subseteq V \setminus \{y\}$$

and

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[d] \arrow[dl, "\cong"'] & \mathbb{Z} \cong H_n(V, V \setminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[u] \arrow[ul, "\cong"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$0 = H_{n-1}(S^n - \{x_i\})$

$H_n(S^n - \{x_i\}) = 0$

Thus, $\exists!\, d \in \mathbb{Z}$ s.t.

$$f_* : \underbrace{H_n(U_i, U_i \setminus \{x_i\})}_{\cong H_n(S^n)} \to \underbrace{H_n(V, V \setminus \{y\})}_{\cong H_n(S^n)}$$

$$f_*(z) = d \cdot z \qquad \forall z \in$$

This number d is called the local degree of f at x_i, denoted by $\deg f|_{x_i}$

Example

Consider

$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{C}$$

Let

$$f : S^1 \to S^1, \quad z \longmapsto z^2$$

and $y = 1 \Rightarrow f^{-1}(y) = \{ \pm 1 \} = \{ x_1 = 1, \ x_2 = -1 \}$

Let

$$V = \{ e^{2\pi i \theta} \mid -\tfrac{\pi}{4} < \theta < \tfrac{\pi}{4} \}$$

$$U_1 = \{ e^{2\pi i \theta} \mid -\tfrac{\pi}{8} < \theta < \tfrac{\pi}{8} \}$$

$$U_2 = \{ e^{2\pi i \theta} \mid \tfrac{7}{8}\pi < \theta < \tfrac{9}{8}\pi \}$$

$\Rightarrow$ $f|_{U_j} : U_j \to V$, $j = 1, 2$, are homeomorphisms

$\Rightarrow$ $$f_* : H_n(U_j, U_j \setminus \{x_j\}) \xrightarrow{\cong} H_n(V, V \setminus \{y\})$$

are isomorphisms.

$\Rightarrow$ $\deg f|_{x_j} = \pm 1$. Q: $+1$ or -1?

$\cup \quad x_j$

$\tilde{\omega}(t, 1-t) = e^{2\pi i \left(\frac{1}{2} + \frac{1}{2}(t - \frac{1}{2})\right)}$

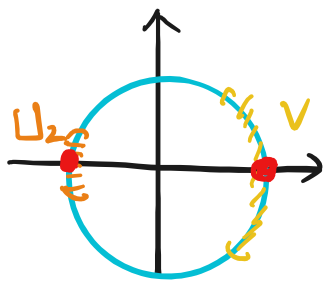
$$\begin{tikzcd}
{[\tilde{\omega}]} \arrow[ddd, leftrightarrow] & H_1(U_2, U_2 - \{-1\}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_1(V, V - \{1\}) \arrow[d, "\cong"] & {[\omega']} \arrow[ddd, leftrightarrow] \\
& H_1(S^1, S^1 - \{-1\}) & H_1(S^1, S^1 - \{1\}) & \\
& & & \\
\langle [\omega] \rangle = & H_1(S^1) \arrow[uu, "\cong"] \arrow[r, "f_*"'] & H_1(S^1) \arrow[uu, "\cong"'] & {[\omega]}
\end{tikzcd}$$

$\underline{H_1(S^1)} \cong \mathbb{Z}$ $\qquad$ $\underline{H_1(S^1)} \cong \mathbb{Z}$

$\omega : \Delta^1 \to S^1, \; (t, 1-t) \mapsto e^{2\pi i t}$

$\Delta^1 = \{(t, 1-t) \in \mathbb{R}^2 \mid t \in [0,1]\}$

$(f \circ \omega) : (t, 1-t) \mapsto e^{\overset{4}{2\pi i}\, \cancel{2} t}$



By "counting direction",

$\deg f|_{x_j} = +1$

#

<u>Prop.</u>

Suppose $f : S^n \to S^n$, $f^{-1}(y) = \{x_1, \cdots, x_m\}$.

Then

$$\deg f = \sum_{j=1}^{m} \deg f|_{x_j}$$

Generators of $H_n(S^n)$

Recall:

The pair $(D^n, \partial D^n = S^{n-1})$ is a good pair.

$\Rightarrow$ we have the long exact seq:

$$\cdots \to \tilde{H}_k(D^n) = 0 \to \tilde{H}_k(D^n, S^{n-1}) \xrightarrow[\partial]{\cong} \tilde{H}_{k-1}(S^{n-1}) \to \tilde{H}_{k-1}(D^n) = 0 \to \cdots$$

$$\tilde{H}_k(D^n, S^{n-1}) \cong \tilde{H}_k(D^n/S^{n-1}) = \tilde{H}_k(S^n)$$

$$\Rightarrow \quad \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) \cong \cdots \cong \tilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z} & k-n=0 \\ 0 & \text{other} \end{cases}$$

Case $n=0$:

$$S^0 = \{x, y\}$$

The singular chain complex of S^0 is:

augmentation

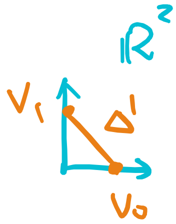
$$\cdots \to \underset{\substack{\| \\ \Delta^1 \to S^0 \\ \cong \\ [0,1]}}{C_1(S^0)} \cong \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\partial_1} \underset{\substack{\cong \\ \mathbb{Z}x \oplus \mathbb{Z}y}}{C_0(S^0)} \xrightarrow{\varepsilon} \mathbb{Z}$$

$$ax+by \overset{\varepsilon}{\longmapsto} a+b$$

$$\ker(\varepsilon) = ax - ay = a(x-y) = \{ax+by \mid a+b=0,\ b=-a\}$$

$$\Rightarrow \quad \tilde{H}_0(S^0) = \langle [x-y] \rangle$$

<u>Case $n=1$</u>:



$\overset{y}{-1}$, $\overset{x}{1}$: $[-1,1] \subseteq \mathbb{R}^1$

Let

$$\sigma : \Delta^1 = [v_0, v_1] \to \underline{D^1} \qquad \partial D^1 = S^0 = \{x, y\}$$

be a homeomorphism s.t. $\quad ([\sigma] \in H_1(D^1, S^0))$

$$\sigma(v_0) = y, \quad \sigma(v_1) = x$$

<u>Claim</u>

$$\partial([\sigma]) = [x-y]$$

$\partial : \tilde{H}_1(D^1, S^0) \to \tilde{H}_0(S^0)$ connecting homomorphism $\cong \mathbb{Z}$

$$\begin{tikzcd}
 & 0 & 0 & 0 \\
\cdots \arrow[r] & C_1(D^1, S^0) \arrow[u] \arrow[r] & C_0(D^1, S^0) \arrow[u] \arrow[r] & 0 \arrow[u] \\
\cdots \arrow[r] & C_1(D^1) \ni \sigma \arrow[u] \arrow[r, "\partial_1"] & C_0(D^1) \arrow[u] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[u] \\
\cdots \arrow[r] & C_1(S^0) \arrow[u] \arrow[r] & C_0(S^0) \arrow[u] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

$\partial[\sigma]$

∂^{-1}

$\sigma(v_1) - \sigma(v_0) = x - y \in C_0(D^1)$

$x - y \in C_0(S^0)$

So $\langle[\sigma]\rangle = \tilde{H}_1(D^1, S^0) \cong \tilde{H}_0(S^0) = \langle[x-y]\rangle$

a generator of

Furthermore, by the iso.

induced by quotient map $D^1 \to D^1/S^0$

$$\tilde{H}_1(D^1, S^0) \cong \tilde{H}_1(D^1/S^0) \cong \tilde{H}_1(S^1)$$

$$[\sigma] \longmapsto [\omega]$$

$(\sigma: \Delta^1 \to D^1 = [-1,1])$

$\omega: \Delta^1 \to S^1 \subseteq \mathbb{C}$,

$\omega(t) = e^{\pi i \sigma(t)}$

This is a generator of $\tilde{H}_1(S^1)$ #