Alg Topo 4/23

Today: Homology groups of CW complexes

They can be computed by "cellular chain complexes" ← much simpler than singular chain complex

Lemma

Suppose X is a CW complex

(i) $$H_k(X^n, X^{n-1}) = \begin{cases} 0, & k \neq n \\ \boxed{\text{free abelian group with basis one-to-one corresponding to the n-cells of } X} & k = n \end{cases}$$

$\cong \mathbb{Z}^{C_n}$

$C_n = \#$ of n-cells of X

(ii) $H_k(X^n) = 0$ for $k > n$

In particular, if X is finite-dimensional, then $H_k(X)=0$ for $k>\dim X$.

(iii) The homomorphism

$$H_k(X^n)\longrightarrow H_k(X)$$

induced by the inclusion map $X^n\hookrightarrow X$ is $\begin{cases}\text{an isomorphism for } k<n\\ \text{surjective for } k=n.\end{cases}$

pf

(i) Since (X^n, X^{n-1}) is a good pair and

$$X^n/X^{n-1}\cong \overbrace{S^n\vee\cdots\vee S^n}^{C_n}$$

we have

$$H_k(X^n,X^{n-1})\cong \tilde{H}_k(X^n/X^{n-1})\cong \tilde{H}_k(\overbrace{S^n\vee\cdots\vee S^n}^{C_n})$$

$$\cong\begin{cases}\mathbb{Z}^{C_n} & k=n\\ 0 & k\neq n\end{cases}$$

(ii) Consider the long exact seq. of (X^n, X^{n-1}):

$$(*)\quad \cdots \to \underbrace{\boxed{H_{k+1}(X^n, X^{n-1})}}_{=0} \to H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n) \to \underbrace{\boxed{H_k(X^n, X^{n-1})}}_{=0} \to \cdots$$

$k>n \Rightarrow k+1>n$ $\qquad\qquad$ $k>n$

So, for $k>n\geq 0$, ($k>0$)

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$

(iii) We prove it for finite-dimensional CW cx here.

For infinite-dimensional case, see p. 138-139.

Assume $X = X^N$.

By (*),

$$H_k(X^{\cancel{n-1}\,n}) \to H_k(X^{\cancel{n}\,n+1}) \text{ is } \begin{cases} 1\text{-}1, & k+1 \neq \cancel{n}\,n+1 \iff k \neq n \\ \text{onto}, & k \neq \cancel{n}\,n+1 \end{cases}$$

So for $k<n \Rightarrow k \neq n, n+1, n+2, \dots$

$$H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \xrightarrow{\cong} H_k(X^{n+2}) \to \cdots \xrightarrow{\cong} H_k(X^N = X)$$

is an isomorphism.

If $k=n$, then

$$H_k(X^n) \xrightarrow{\text{onto}} H_k(X^{n+1}) \xrightarrow{\cong} H_k(X^{n+2}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^N = X)$$

is surjective. #

Let X be a CW complex. By Lemma,

$$H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{c_n}$$

where $c_n = \#$ of n-cells of X.

Define

$$\begin{tikzcd} d_n = j_* \circ \partial : H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[d, phantom, "\cong" rotate=90] & H_{n-1}(X^{n-1}) \arrow[r, "j_*"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z}^{c_n} \arrow[rr, "d_n"'] & & \mathbb{Z}^{c_{n-1}} \end{tikzcd}$$

<u>Remark</u>

We used the long exact seq's of

$$(X^n, X^{n-1}) \quad \text{and} \quad (X^{n-1}, X^{n-2})$$

$$\begin{tikzcd} & H_{n-1}(X^n) \arrow[r] & \cdots & \vdots \\ & & & H_{n-2}(X^{n-2}) \arrow[u] \\ H_n(X^n, X^{n-1}) \arrow[r, "\partial"] & H_{n-1}(X^{n-1}) \arrow[uu, "i_*"] \arrow[rr, "j_*"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[u, "\partial"] \end{tikzcd}$$

$$\begin{tikzcd}
 & H_n(X^n, X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}, X^{n-2}) \\
\cdots \arrow[r] & H_n(X^n) \arrow[u, "j_*"] & \cdots \arrow[r] & H_{n-1}(X^{n-2}) \arrow[ul, "i_*"]
\end{tikzcd}$$

Lemma

$$d_n \circ d_{n+1} = 0$$

pf

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r] \arrow[rr, bend left, "d_{n+1}"] & H_n(X^n) \arrow[r] \arrow[rr, bend left, "0"] & H_n(X^n, X^{n-1}) \arrow[r] \arrow[dr, bend right, "d_n"] & H_{n-1}(X^{n-1}) \arrow[d] \\
 & & & H_{n-1}(X^{n-1}, X^{n-2})
\end{tikzcd}$$

$$= 0$$

#

Def

The chain complex

$$\left(\underbrace{H_n(X^n, X^{n-1})}_{\cong\, \mathbb{Z}^{C_n},\ C_n = \#\text{ of } n\text{-cell of } X},\ d_n \right)$$

is called the cellular chain complex of the CW complex X.

Its induced homology is called the

cellular homology of X, temporarily denoted by $\boxed{H_n^{CW}(X)}$. $= \ker(d_n) / \mathrm{im}(d_{n+1})$

<u>Thm</u> (Thm 2.35)

If X is a CW complex, then

$$H_n^{CW}(X) \cong H_n(X). \qquad \forall n$$

pf

(*)

$$\begin{tikzcd}
 & & & H_n(X) \arrow[d, phantom, "\cong"] & & & & \\
 & & & H_n(X^{n+1}) \arrow[r] & H_n(X^{n+1}, X^n) = 0 & & & \\
 & & 0 = H_n(X^{n-1}) \arrow[r] & H_n(X^n) \arrow[u, "i_*", "\text{onto}"'] \arrow[dr, "j_*", "1\text{-}1"'] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\partial"] \arrow[rrr, "d_{n+1}"] & & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
0 = H_{n+1}(X^n) \arrow[r] & H_{n+1}(X^{n+1}) \arrow[u, "1\text{-}1"] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*", "1\text{-}1"'] & & \\
 & & & & & 0 = H_{n-1}(X^{n-2}) \arrow[u] & &
\end{tikzcd}$$

$\Downarrow$

$(j_* \partial)(x) = 0$

$\Leftrightarrow \partial(x) = 0$

So $\ker(\partial) = \ker(j_* \partial)$

$$X^n \overset{i}{\hookrightarrow} X$$

Since (*)

$$H_n(X^n) \xrightarrow{i_*} H_n(X) \quad \text{is onto}$$

we have

$$H_n(X) \cong H_n(X^n) / \ker(i_*) = H_n(X^n) / \partial(H_{n+1}(X^{n+1}, X^n))$$

$$\frac{\ker(i_*)}{\partial(H_{n+1}(X^{n+1}, X^n))}$$

$$\cong \quad (\because j_* \text{ is } 1\text{-}1) \quad \frac{\boxed{j_*(H_n(X^n))}}{\boxed{j_*(\partial}(H_{n+1}(X^{n+1}, X^n)))}$$

$$j_*(H_n(X^n)) = \ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})) = \ker(j_*\partial) = \ker(d_n)$$

$$j_*\partial = d_{n+1}$$

$$= \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})} = H_n^{CW}(X) \qquad \#$$

Cor

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise} \end{cases}$$

pf

Recall

$$\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$$

Thus, the cellular chain complex of $\mathbb{CP}^n$ is

$$\dots \to 0 \to \mathbb{Z}e^{2n} \xrightarrow{d_{2n} = 0} 0 \xrightarrow{d_{2n-1} = 0} \mathbb{Z}e^{2n-2} \xrightarrow{d_{2n-2} = 0} \dots \to 0 \to \mathbb{Z}e^0$$

So

$$H_k(\mathbb{C}P^n) = \begin{cases} \ker(d_{2m}) / \operatorname{im}(d_{2m+1}) = \mathbb{Z}/0 \cong \mathbb{Z}, & k = 2m, \ 0 \le m \le n \\ 0/0 = 0 & \text{otherwise} \end{cases}$$

#

Remark

① If X has NO n-cells, then $H_n(X) = 0$.

$\Rightarrow H_n(X^n, X^{n-1}) \cong \mathbb{Z}^k \supseteq \ker(d_n)$

?② If X has k n-cells, then $\underbrace{H_n(X)}_{\ker(d_n)/\operatorname{im}(d_{n+1})}$ can generated by k elements.

③ If X has k n-cells, zero $(n-1)$-cell and zero $(n+1)$-cell, then $H_n(X) \cong \mathbb{Z}^k$ e.g. $\mathbb{C}P^n$

$$\to 0 \xrightarrow{d_{n+1}} \mathbb{Z}^k \xrightarrow{d_n} 0 \to$$

Cellular boundary formula

One can compute d_n by "degrees"

Def

Let $f: S^n \to S^n$, $n > 0$.

Since $H_n(S^n) \cong \mathbb{Z}$, $\exists!\, d \in \mathbb{Z}$ s.t

$$f_*: H_n(S^n) \longrightarrow H_n(S^n)$$

$$f_*(x) = d \cdot x \qquad \forall\, x \in H_n(S^n)$$

This number d is called the degree of f, denoted by $\deg(f)$

Remark

$n=1$: $\deg(f: S^1 \to S^1)$ = winding number of f

(We will prove it later)

e.g. $S^1 = \{ z \in \mathbb{C} \mid |z|=1 \}$

$\deg(z^k) = k$.

Prop (p 134)

Let $f, g : S^n \to S^n$.

(i) $\deg(\mathrm{id}_{S^n}) = 1$

(ii) If f is NOT surjective, then

$$\deg(f) = 0.$$

(iii) If $f \simeq g$, then $\deg(f) = \deg(g)$

(For the converse, see Cor 4.25)

(iv) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$

<u>pf of (ii)</u>

If $f : S^n \to S^n$ is not onto, then $\exists\, x_0 \in S^n \setminus f(S^n)$

Let $\tilde{f}$ be the corestriction

$$\tilde{f} : S^n \to S^n - \{x_0\}, \quad \tilde{f}(x) = f(x)$$

and

$$i : S^n \setminus \{x_0\} \hookrightarrow S^n$$

be the inclusion map.

$\Rightarrow$
$$f = i \circ \tilde{f}$$

$\Rightarrow$
$$f_* = H_n(S^n) \xrightarrow{\tilde{f}_*} \overset{0}{\overset{\parallel}{\boxed{H_n(S^n \setminus \{x_0\})}}} \xrightarrow{i_*} H_n(S^n)$$

$$= 0$$

$\Rightarrow$ $\deg(f) = 0$ #

<u>exer</u>

Study the other properties about degrees

... properties about degrees
in p. 134-137.

Let X be a CW complex with n-cells e_α^n.

$\Rightarrow$ $$H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z} e_\alpha^n = \text{free abelian gp with basis } \{e_\alpha^n\}_\alpha$$

<u>Thm</u> (p. 140)

The cellular boundary homomorphism

$$d_n : \underbrace{H_n(X^n, X^{n-1})}_{\text{basis } \{e_\alpha^n\}} \longrightarrow \underbrace{H_{n-1}(X^{n-1}, X^{n-2})}_{\text{basis } \{e_\beta^{n-1}\}}, \quad n>1.$$

is given by the formula

Acually, it is a finite sum

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1}$$

where $d_{\alpha\beta}$ is the degree of

$$S^{n-1} = \partial D_\alpha^n \xrightarrow[\text{map}]{\text{attaching}} X^{n-1} \xrightarrow[\text{map}]{\text{quotient}} X^{n-1} / X^{n-1} - e_\beta^{n-1} \cong S^{n-1}$$

Furthermore,

$\ker\left(C^1(X)/C(X^0) \to C^0(X^1)/C^0(X)\right)$ $\quad$ im(...)

$$d_1 : \underline{H_1(X^1, X^0)} \to H_0(X^0)$$

$$d_1([\gamma]) = \gamma(1) - \gamma(0)$$

where
$$\gamma : \Delta^1 \cong [0,1] \longrightarrow X^1$$
is any path with $\gamma(0), \gamma(1) \in X^0$

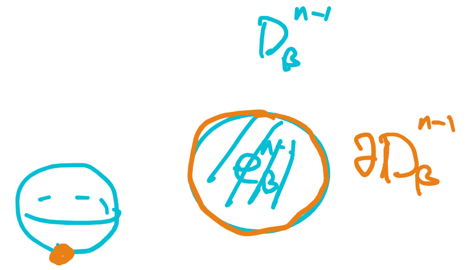
pf

Consider

"characteristic map"

- $\Phi_\alpha : D_\alpha^n \hookrightarrow X^{n-1} \sqcup_\alpha D_\alpha^n \twoheadrightarrow X^n \to X$

  $\varphi_\alpha : \Phi_\alpha|_{\partial D_\alpha^n}$



- $\tilde{q} : X^{n-1} \longrightarrow X^{n-1}/X^{n-2}$

- $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S_\beta^{n-1} \cong D_\beta^{n-1}/\partial D_\beta^{n-1}$

$$\begin{tikzcd}
X^{n-1}/X^{n-2} \arrow[dr] & & D_\beta^{n-1}/\partial D_\beta^{n-1} \arrow[dl, "\cong"'] \\
& X^{n-1}/(X^{n-1}-e_\beta^{n-1}) & \\
& X^{n-1} \arrow[u] & D_\beta^{n-1} \arrow[uu] \arrow[l, "\Phi_\beta"]
\end{tikzcd}$$

- $\Delta_{\alpha\beta} : S_\alpha^{n-1} = \partial D_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\tilde{q}} X^{n-1}/X^{n-2} \xrightarrow{q_\beta} S_\beta^{n-1}$

We have the commutative diagram:

"$[D_\alpha^n]$" $\longrightarrow$ "e_α^n" $\longmapsto$ $d_{\alpha\beta}\, e_\beta^{n-1}$

$$
\begin{tikzcd}
{[e_\alpha]} \arrow[d, mapsto] & H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial"', "\cong"] \arrow[d, "\Phi_{\alpha *}"] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) & e_\beta \\
e^n_\alpha & H_n(X^n, X^{n-1}) \arrow[r, "\partial"'] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "\tilde{q}_*"] \arrow[d, "q_*"'] & \tilde{H}_{n-1}(\underbrace{X^{n-1}/X^{n-2}}_{\vee_\beta S^{n-1}_\beta}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] & \cong \bigoplus_\beta \mathbb{Z} e^{n-1}_\beta \\
 & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}) &
\end{tikzcd}
$$

$\Rightarrow$

$$
d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta \qquad \#
$$