Alg Topo 4/16

CW complex

Def (page 5, 519)

A CW complex (or cell complex) is a space X constructed in the following way:

(i) Start with a decrete set X^0, called the 0-cells of X

(ii) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$$

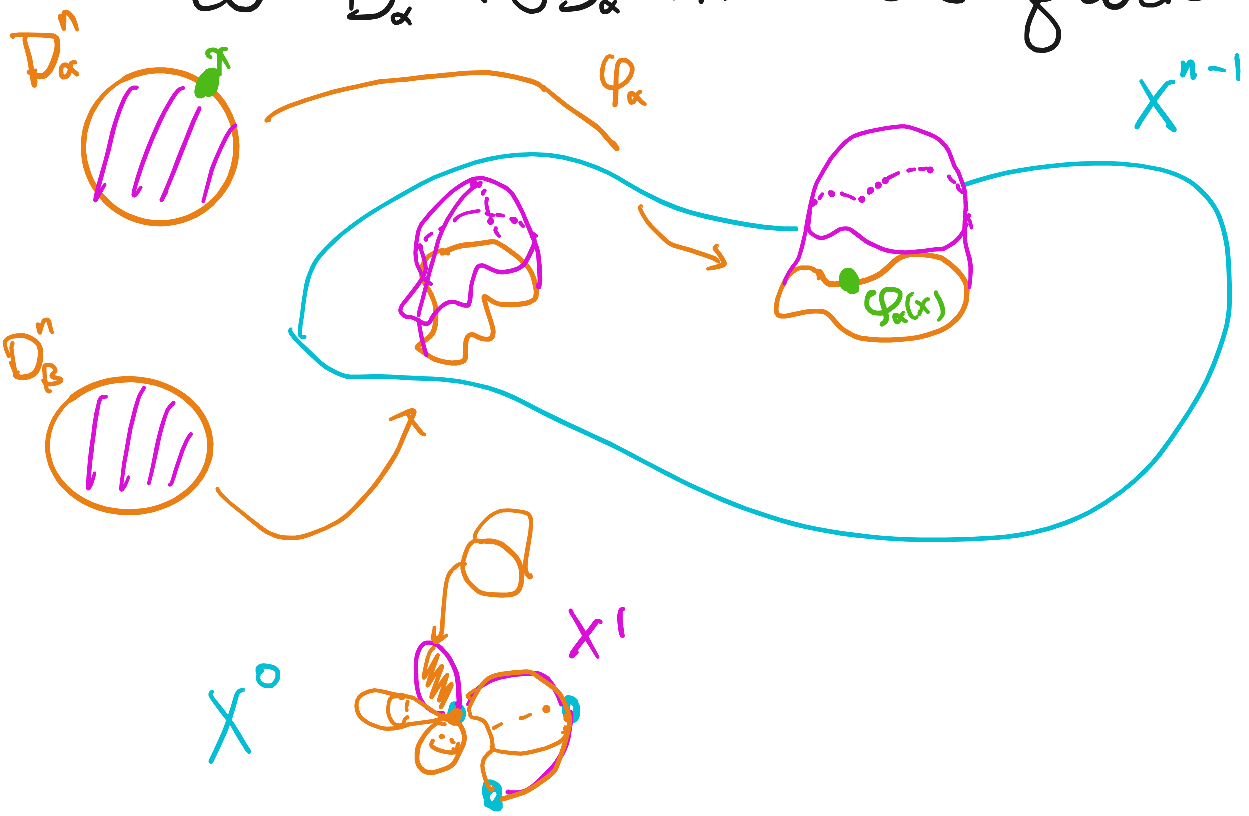
That is,

X^n … $X^{n-1} \sqcup \coprod D^n /$

$$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha \Big/ x \sim \varphi_\alpha(x),\ \underline{x \in \partial D^n_\alpha}$$

where
$$D^n_\alpha = D^n.$$

Note that the cell e^n_α is homeomorphic to $D^n_\alpha \setminus \partial D^n_\alpha$ under the quotient map



(iii) The space $X = \bigcup_n X^n$ is equipped with the weak topology:

A subset $A \subseteq X$ is open iff

$A \cap X^n$ is open for each n.

For each cell e_α^n, the map

$$\Phi_\alpha : D^n = D_\alpha^n \hookrightarrow (X^{n-1} \amalg \coprod_\alpha D_\alpha^n) \twoheadrightarrow X^n \hookrightarrow X$$

is called the characteristic map of e_α^n

If $X = X^n$ for some n, then X is said to be finite-dimensional. In this case, the smallest number n s.t.

$$X = X^n$$

is called the dimension of X.

Remark (page 520)

"CW" means the following properties:

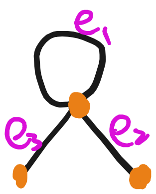
(i) C = closure-finiteness:

The closure of each cell meets only finitely many other cells.

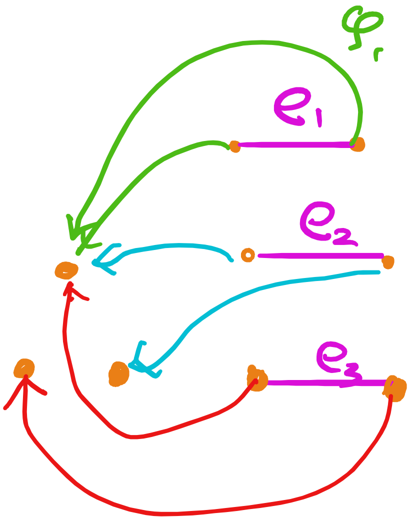
(iii) W = weak topology:

A set is closed iff its intersection with the closure of each cell is closed.

<u>Example</u>

$X =$ 

$= X^1$

$X^0 =$ 

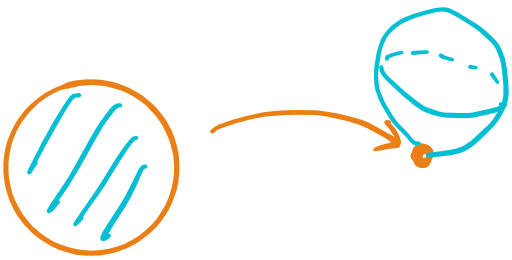
A one-dimensional CW complex.

<u>Example</u>

The n-sphere S^n has a CW structure with 2 cells: e^0 and e^n

$$X^0 = \bullet = X^1 = \cdots = X^{n-1}$$

$$\partial D^{n-1} \cong S^{n-1} \xrightarrow{\varphi} \bullet$$

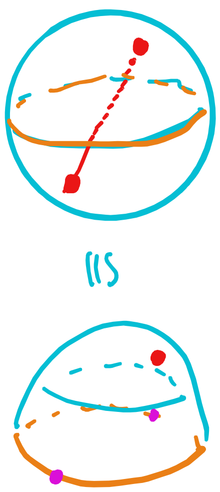


Example (Example 0.4, 0.5)

The real projective n-space $\mathbb{RP}^n$ is

$$\mathbb{RP}^n = \{\text{one-dimensional real vector spaces in } \mathbb{R}^{n+1}\}$$

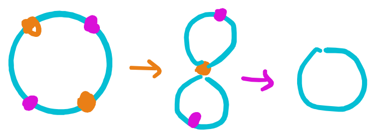
$$= \mathbb{R}^{n+1} \setminus \{\vec{0}\} \Big/ \vec{v} \sim \lambda \vec{v},\ \lambda \neq 0,\ \vec{v} \in \mathbb{R}^{n+1} \setminus \{0\}$$



$$= S^n \Big/ \vec{v} \sim -\vec{v}$$

Note

$\mathbb{RP}^1 = S^1 / \vec{v} \sim -\vec{v} \cong S^1$



$$= D^n \Big/ \vec{v} \sim -\vec{v},\ \vec{v} \in \partial D^n$$

$\partial D^n \cong S^{n-1}$

Thus,

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\varphi_n} e^n.$$

where

$$\varphi_n = \text{quotient map}: \partial D^n = S^{n-1} \to \mathbb{RP}^{n-1} = S^{n-1}/_{\vec{v} \sim -\vec{v}}$$

So

$$\boxed{\mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} e^n}$$

— an n-dimensional CW complex

In fact, one can consider

$$\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots$$

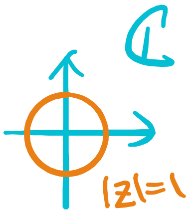
Example (Example 0.6)

The complex projective n-space $\mathbb{CP}^n$ is

$$\mathbb{CP}^n = \left\{ \begin{array}{l} \text{one-dimensional complex} \\ \text{vector subspaces in } \mathbb{C}^{n+1} \end{array} \right\}$$

$$= \overset{\mathbb{R}^{2n+2} \cong}{\mathbb{C}^{n+1}} \setminus \{\vec{0}\} \Big/ \vec{v} \sim \lambda \vec{v},\ \lambda \neq 0,\ \lambda \in \mathbb{C},\ \vec{v} \in \mathbb{C}^{n+1} \setminus \{\vec{0}\}$$

$\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid |z_1|^2 + \cdots + |z_{n+1}|^2$



$$= \dfrac{S^{2n+1}}{\vec{v} \sim \lambda \vec{v},\ |\lambda| = 1,\ \lambda \in \mathbb{C}}$$

$$S^{2n+1} = \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C} \mid \ldots = 1 \}$$

<u>Notation</u>:

$$[z_1 : z_2 : \cdots : z_{n+1}] \in \mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} \big/ \sim$$

= the equivalence class represented by $(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{\vec{0}\}$

$\sim (\lambda z_1, \ldots, \lambda z_{n+1})$ hope > 0

Let $(z_1, \cdots, z_{n+1}) \in S^{2n+1} \subseteq \mathbb{C}^{2n+1}$.

① If $z_{n+1} \neq 0$, then

$$(\lambda z_1, \cdots, \lambda z_n, \lambda z_{n+1} = |z_{n+1}| > 0),$$

$\lambda = \dfrac{|z_{n+1}|}{z_{n+1}}$, is the unit vector with a positive last component that is equivalent to $(z_1, \ldots, z_{n+1})$ in $\mathbb{CP}^n$.

So

$S = \ldots \in \mathbb{CP}^n \mid z \ldots \neq 0 \}$

$$\{[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n \mid z_{n+1} \neq 0\}$$

$$= \{[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n \mid z_{n+1} > 0\}$$

$[w_1 : \cdots : w_n : \sqrt{1-\|\vec{w}\|^2}]$ $\qquad$ $[z_1 : \cdots : z_n : z_{n+1}]$, $z_{n+1} > 0$

$$\updownarrow \qquad \cong \; \underline{D^{2n} \setminus \partial D^{2n}} \subseteq \mathbb{C}^n \qquad \downarrow$$

$(w_1, \cdots, w_n) = \vec{w} \in$ $\qquad$ $(z_1, \cdots, z_n)$

$|z_1|^2 + \cdots + |z_n|^2 < 1$ $\qquad$ $+|z_{n+1}|^2 = 1$

② If $z_{n+1} = 0$, then

$$\{[z_1 : \cdots : z_n : z_{n+1}] \in \mathbb{CP}^n \mid z_{n+1} = 0\}$$

$z_{n+1} = 0$

$$\cong \mathbb{CP}^{n-1}$$

So

$$\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup_\varphi e^{2n}$$

$$\Rightarrow \boxed{\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}} \quad \#$$

<u>Def</u>

A subcomplex of a CW complex X is a closed subspace $A \subseteq X$ that is a union of cells of X.

A pair (X, A) of a CW complex X and a subcomplex is called a CW pair

exer

A CW pair is a good pair.

Cor

If a CW complex X is a union of subcomplexes A and B, then

$$(B, A\cap B) \hookrightarrow (X, A)$$

good pair ⇐ $\sim (X, U)$ for some nbd U of A

induces isomorphisms

$$H_n(B, A\cap B) \cong H_n(X, A) \quad \forall n.$$

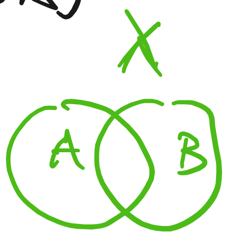
$H_n(X,A) \cong \tilde{H}_n(X/A)$

pf

Since CW pairs are good pairs,

$$\begin{tikzcd}
H_n(B, A\cap B) \arrow[r, "\cong"] \arrow[d, "\text{Cor}"'] & H_n(B/A\cap B, A\cap B/A\cap B) \arrow[d] & \tilde{H}_n(B/A\cap B) \arrow[l, "\cong"'] \arrow[d, "\text{ILS}"] \\
H_n(X, A) \arrow[r, "\cong"'] & H_n(X/A, A/A) & \tilde{H}_n(X/A) \arrow[l, "\cong"']
\end{tikzcd}$$

$\because B/A\cap B \cong X/A$

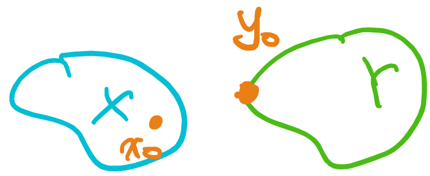


#

Def

Given spaces X and Y with $x_0 \in X$, $y_0 \in Y$, the wedge sum $X \vee Y$ is

$$X \vee Y = X \sqcup Y \big/ x_0 \sim y_0$$



If we are given X_α, $x_\alpha \in X_\alpha$, $\alpha \in \Lambda$, $X \vee Y$

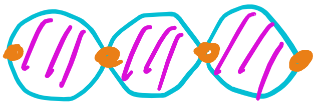
then we have

$$\bigvee_{\alpha\in\Lambda} X_\alpha = \coprod_{\alpha\in\Lambda} X_\alpha \Big/ x_\alpha \sim x_\beta,\ \alpha,\beta \in \Lambda$$

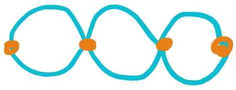
Example

Given X is the CW complex

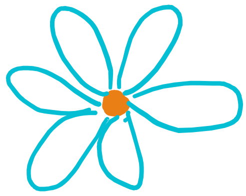
Suppose X is CW complex

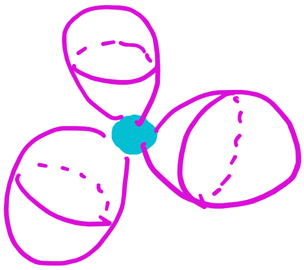
$X =$  $= X^2$

$X^0 =$ 

$X^1 =$ 

$\Rightarrow$

$X^1 / X^0 =$  $\cong \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{6 \text{ times}}$

$X^2 / X^1 =$  $\cong S^2 \vee S^2 \vee S^2$

In general,

$$X^n / X^{n-1} = \underbrace{S^n \vee \cdots \vee S^n}_{k \text{ times}}$$

$k =$ number of n-cells in X

<u>Prop</u>

For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions

$$i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$$

induce an isomorphism

$$\bigoplus_\alpha i_\alpha : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n(\vee_\alpha X_\alpha)$$

provided the pairs $(X_\alpha, \{x_\alpha\})$ are good.

pf

$$\vee_\alpha X_\alpha \cong \coprod_\alpha X_\alpha / \coprod_\alpha \{x_\alpha\}$$

$$\tilde{H}_n(\vee_\alpha X_\alpha) \cong \tilde{H}_n\left(\coprod_\alpha X_\alpha / \coprod_\alpha \{x_\alpha\}\right)$$

good pairs assumption $\cong \tilde{H}_n\left(\coprod_\alpha X_\alpha , \coprod_\alpha \{x_\alpha\}\right)$

$$\cong \bigoplus_\alpha \tilde{H}_n(X_\alpha, \{x_\alpha\})$$

$$\cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

exact seq:

$$\cdots \to \tilde{H}_n(\{x_\alpha\}) \to \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n(X_\alpha, \{x_\alpha\}) \to \tilde{H}_{n-1}(\{x_\alpha\}) \to \cdots$$

$\tilde{H}_n(\{x_\alpha\}) = 0$, $\tilde{H}_{n-1}(\{x_\alpha\}) = 0$, $\forall$

<u>Next:</u>

Assume X is a CW complex

The singular homology of X actually

can be computed by a smaller chain complex, called the cellular chain complex

$$\cdots \to H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

$$H_n(X^n, X^{n-1}) \cong \tilde{H}_n(X^n/X^{n-1}) \cong \tilde{H}_n(S^n \vee \cdots) \cong \mathbb{Z} \oplus \cdots \cong \mathbb{Z}^{k_n}$$

k_n = # of cells in X